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Some New Conditions for Weighted Fourier Inequality

[1*]Sokhobiddin Akhatkulov and [2]Mohamed Abubakar

[1]School of Mathematical Sciences,
Faculty of Science and Technology, University Kebangsaan Malaysia,
43600 UKM Bangi, Selangor, Malaysia

[2]Department of Mathematics, Faculty of Science and
Institute for Mathematical Research, Universiti Putra Malaysia,
43400 UPM Serdang, Selangor, Malaysia

E-mail: akhatkulov@yahoo.com and wiilow29@hotmail.com

*Corresponding author

ABSTRACT

In this paper we give some new conditions on weights u and v for weighted Fourier transform norm inequality $||\hat{f}||_{q,u} \leq C||f||_{p,v}$ in case $1 < p,$ q $< \infty$.

Keywords: Weight function, weighted Fourier transform norm inequality, weighted Lebesgue spaces, Hardy inequality, decreasing rearrangement.

1. INTRODUCTION

The Fourier transform of a complex-valued Lebesgue measurable $f: \mathbb{R}^n \to \mathbb{C}$ function on the Euclidean space $\mathbb{R}^n$ is formally defined as

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma)$$

$$= \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\cdot\gamma} d\boldsymbol{x} \qquad (1)$$

where $\gamma \in \widehat{\mathbb{R}}^n (= \mathbb{R}^n)$ is a spectral variable and $\boldsymbol{x} \in \mathbb{R}^n$ is a space variable. If f belongs to the Lebesgue space $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then its L^p norm is designated $||f||_p$. It is elementary to see that $||\hat{f}||_\infty \leq ||f||_1$ for $f \in$

$L^1(\mathbb{R}^n)$; and if $f \in L^2(\mathbb{R}^n)$ then the Plancherel theorem asserts that $||\hat{f}||_2 = ||f||_2$. Both of these norm relationships can be viewed as special cases of the weighted Fourier transform norm inequality

$$||\hat{f}||_{q,u} \leq C||f||_{p,v} \tag{2}$$

where

$$||\hat{f}||_{q,u} = \left(\int_{\widehat{\mathbb{R}}^n} \left|\hat{f}(\gamma)\right|^q u(\gamma)d\gamma\right)^{1/q}$$

and

$$||f||_{p,v} = \left(\int_{\mathbb{R}^n} |f(\boldsymbol{x})|^p v(\boldsymbol{x})d\boldsymbol{x}\right)^{1/p},$$

where u and v are non-negative, locally integrable functions on $\widehat{\mathbb{R}}^n$ and $\mathbb{R}^n$ respectively and C is a constant independent of f. By definition, $L_v^p(\mathbb{R}^n)$ is the space of complex-valued Lebesgue measurable functions $f: \mathbb{R}^n \to \mathbb{C}$ for which $||f||_{p,v} < \infty$. The relation (2) reminds us the following well-known Hardy inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{1/q} \leq C_{p,q}\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{3}$$

with parameters p, q such that $1 < p, q < \infty$ and given weight functions $u \geq 0$, $v \geq 0$. Here $f \geq 0$ is a measurable function on $(0, \infty)$. For the case $1 < p \leq q < \infty$ a necessary and sufficient condition on the weights $u \geq 0$, $v \geq 0$ for (3) to hold for all $f \geq 0$ is either the well-known Muckenhoupt condition (see Opic and Kufner (1990) and Kufner and Persson (2003))

$$A_M \coloneqq \sup_{0<x<\infty}\left(\int_x^\infty u(t)dt\right)^{1/q}\left(\int_0^x v^{1-p'}(t)dt\right)^{1/p'} < \infty$$

or the following two alternatives which can be found in Kufner and Persson (2003):

$$A_{PS}^1 \coloneqq \sup_{0<x<\infty}\left(\int_0^x \left(\int_0^t v^{1-p'}(\tau)d\tau\right)^q u(t)dt\right)^{1/q}\left(\int_0^x v^{1-p'}(t)dt\right)^{-1/p} < \infty$$

or

$$A_{PS}^2 := \sup_{0<x<\infty} \left(\int_x^\infty \left(\int_t^\infty u(\tau)d\tau \right)^{p'} v^{1-p'}(t)dt \right)^{1/p'} \left(\int_x^\infty u(t)dt \right)^{-1/q'} < \infty$$

where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. Moreover, for the best constant $C_{p,q}$ in (3) it yields that $C_{p,q} \approx A_M \approx A_{PS}^1 \approx A_{PS}^2$.

Besides conditions given above, it has been taken a number of conditions characterizing inequality (3). For example, some of them are given by Tomaselli, by Wedestig. Moreover, some new conditions can be found in Gogatishvili and *et al.* (2007) and Gogatishvili and *et al.* (2003).

For the case $1 < q < p < \infty$ the inequality (3) is usually characterized by the Maz'ya-Rozin and by Persson-Stepanov conditions. In this case it is known from Kufner *et al.* (2007) and Kufner and Persson (2003), that (3) holds for some finite constant $C_{p,q} > 0$, if and only if one of the following quantities is finite, the Maz'ya-Rozin condition

$$B_{MR} := \left(\int_0^\infty \left(\int_x^\infty u(t)dt \right)^{r/p} \left(\int_0^x v^{1-p'}(t)dt \right)^{r/p'} u(x)dx \right)^{1/r} < \infty$$

or the Persson-Stepanov condition:

$$B_{PS} := \left(\int_0^\infty \left(\int_0^x \left(\int_0^t v^{1-p'}(\tau)d\tau \right)^q u(t)dt \right)^{\frac{r}{p}} \left(\int_0^x v^{1-p'}(t)dt \right)^{q-r/p} u(t)dt \right)^{1/r} < \infty$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and for the best constant $C_{p,q}$ in (3) holds $C_{p,q} \approx B_{MR} \approx B_{PS}$. Moreover, these conditions are not unique, and can be replaced by new equivalent conditions depending on some parameter (see Persson et al. (2007) and Kufner and Kuliev (2010)).

For the Fourier inequality (2), in weighted Lebesgue spaces, this problem was solved by Benedetto and Heinig in Benedetto and Heinig (2003). They derived the following sufficient conditions to ensure the validity of relation (2) similar to Muckenhoupt and Maz'ya-Rozin conditions. For the case $1 < p \leq q < \infty$, this condition reads as

$$A := \sup_{x>0} \left(\int_0^{1/x} u^*(t)dt \right)^{1/q} \left(\int_0^{x} [(1/v)^*(t)]^{p'-1} dt \right)^{1/p'} < \infty$$

where w^* denotes the decreasing rearrangement of the function w (see Definition 2.1, below). And for the case $1 < q < p < \infty$, corresponding condition reads as

$$B := \left(\int_0^{\infty} \left(\int_0^{\frac{1}{x}} u^*(t)dt \right)^{\frac{r}{q}} \left(\int_0^{x} [\left(\frac{1}{v}\right)^* (t)]^{p'-1} dt \right)^{\frac{r}{q'}} [(1/v)^*(x)]^{p'-1} dx \right)^{1/r}$$

$$< \infty$$

Moreover, the best constant C in (2) satisfies

$$C \leq A \begin{cases} (q')^{1/p'} q^{1/q} & if \quad 1 < p \leq q, q \geq 2 \\ p^{\,1/q} (p')^{1/p'} & if \quad 1 < p \leq q < 2 \end{cases}$$

and

$$C \leq B q^{\,1/q} (p')^{1/q'} \quad if \quad 1 < q < p < \infty$$

Our aim in this paper is to give new sufficient conditions for satisfying (2) in weighted Lebesgue spaces for cases $1 < p \leq q < \infty$ and $1 < q < p < \infty$. For this first we give some preliminaries which we will use to formulate the main results and in their proofs in the next sections.

2. PRELIMINARIES

Let us first give the following important definition.

Definition 2.1. Let $(\mathbb{X}; \mu)$ be a measurable space, where $\mathbb{X} \subset \mathbb{R}^n$, and let f be a complex-valued μ-measurable function on $\mathbb{X}$. The distribution function $D_f = [0, \infty) \to [0, \infty)$ of f is defined as

$$D_f(\theta) = \mu\{\boldsymbol{x} \in \mathbb{X}: |f(\boldsymbol{x})| > \theta\}.$$

Two measurable functions f and g on measure spaces $(\mathbb{X}; \mu)$ and $(\mathbb{Y}; \nu)$ respectively are equimeasurable if $D_f = D_g$. The decreasing rearrangement of f on $(\mathbb{X}; \mu)$ is the function $f^*: [0, \infty) \to [0, \infty)$ defined by

$$f^*(t) = inf\{\theta \geq 0: D_f(\theta) \leq t\}.$$

We use the convention inf $\emptyset = \infty$, so that if $D_f(s) > t$ for all $s \in [0, \infty)$ then $f^*(t) = \infty$. For a given μ-measurable f on $(\mathbb{X}; \mu)$, f^* is a non-negative, non-increasing, right continuous function on $[0, \infty)$ and f and f^* are equimeasurable, where f^* is considered as a Lebesgue measurable function on $[0, \infty)$. Furthermore, for any $p \in [0, \infty)$,

$$\int_{\mathbb{X}} |f(\boldsymbol{x})|^p d\mu(\mathbf{x}) = \mathrm{p} \int_0^\infty \mathrm{s}^{\mathrm{p}-1} \mathrm{D_f(s)ds} = \int_0^\infty f^*(t)^{\mathrm{p}} \,\mathrm{dt}.$$

We use the following theorems to derive our main results. These theorems have been obtained by Jodeit and Torchinsky (2007) and by Benedetto and Heinig (2003).

Theorem 2.2. Let $q \geq 2$. Then there is $K_q \geq 0$ such that, for all $f \in L^1 + L^2$ and for all $s > 0$, the inequality

$$\int_0^s \hat{f}^*(t)^q dt \leq \left(K_q\right)^q \int_0^s \left(\int_0^{1/t} f^*\right)^q dt$$

holds, where $L^1 + L^2 = \{f = f_1 + f_2: f_1 \in L^1(\mathbb{R}^n) \; and \; f_2 \in L^2(\mathbb{R}^n)\}$.

Theorem 2.3. Let u and v be weight functions on $\mathbb{R}^n$, suppose $1 < p, \mathrm{q} < \infty$ and let K be a constant from Theorem 2.2 associated with the relevant index great or equal than two. Then there is a constant $\mathrm{C} > 0$ such that, for all $f \in L_v^p(\mathbb{R}^n)$ the inequality

$$\left(\int_{\widehat{\mathbb{R}}^n} \left|\hat{f}(\gamma)\right|^q u(\gamma) d\gamma\right)^{\frac{1}{q}} \leq KC\left(\int_{\mathbb{R}^n} |f(\boldsymbol{x})|^p \, v(\boldsymbol{x}) d\boldsymbol{x}\right)^{\frac{1}{p}} \tag{4}$$

holds in the following ranges and with the following hypotheses on u and v:

(a) if $1 < p \leq q < \infty$ and

$$A := \sup_{x>0} \left(\int_0^{\frac{1}{x}} u^*(t)dt\right)^{\frac{1}{q}} \left(\int_0^x [\left(\frac{1}{v}\right)^* (t)]^{p'-1} dt\right)^{\frac{1}{p'}} < \infty \tag{5}$$

(b) if $1 < q < p < \infty$ and

$$B := \left(\int_0^\infty \left(\int_0^{1/x} u^*(t)dt\right)^{r/q} \left(\int_0^x [(1/v)^*(t)]^{p'-1}dt\right)^{r/q'} [(1/v)^*(x)]^{p'-1}dx\right)^{1/r} < \infty \tag{6}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Moreover, the best constant C in (4) satisfies

$$C \leq A \begin{cases} (q')^{1/p'} q^{1/q} & if \quad 1 < p \leq q, q \geq 2 \\ p^{\,1/q} (p')^{1/p'} & if \quad 1 < p \leq q < 2 \end{cases}$$

and

$$C \leq Bq^{\,1/q} (p')^{1/q'} \quad if \quad 1 < q < p < \infty.$$

Note that by substitution $x = 1/t$, we have

$$\int_0^{1/x} u^*(t)dt = \int_x^\infty \frac{u^*(1/t)}{t^2} dt\,.$$

.

Then condition (5) can be rewritten as

$$A := \sup_{x>0} \left(\int_x^\infty \frac{u^*(1/t)}{t^2} dt\right)^{1/q} \left(\int_0^x [(1/v)^*(t)]^{p'-1} dt\right)^{1/p'} < \infty.$$

This formula reminds the well-known Muckenhoupt condition

$$A_M := \sup_{x>0} \left(\int_x^\infty w(t)dt\right)^{1/q} \left(\int_0^x z^{1-p'}(t)dt\right)^{1/p'} < \infty$$

for the pair of weights w and z, for the Hardy inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q w(x)dx\right)^{1/q} \leq C_{p,q}\left(\int_0^\infty f^p(x)z(x)dx\right)^{1/p} \tag{7}$$

$(f \geq 0)$ for the case $1 < p \leq q < \infty$. Here it is $w(t) = \frac{u^*(1/t)}{t^2}$ and $z(t) = [(1/v)^*(t)]^{-1}$. Analogously, condition (6) corresponds to the Maz'ya-Rozin condition

$$B_{MR} := \left(\int_0^\infty \left(\int_x^\infty w(t)dt \right)^{r/p} \left(\int_0^x z^{1-p'}(t)dt \right)^{r/p'} w(x)dx \right)^{1/r} < \infty$$

for the Hardy inequality (7), for the case $1 < q < p < \infty$. Therefore, it is natural to use our knowledge about the Hardy inequality (7) also for the investigation of the Fourier inequality (4), and we obtain new conditions for u, v. We will use the following equivalence theorem in the proof of our main result. For the proof see Gogatishvili (2007).

Theorem 2.4. For $-\infty < a < b < \infty$, α, β and s positive numbers and f, g, h are measurable, positive functions in (a, b), let

$$F(x) = \int_x^b f(t)dt\,, \qquad G(x) = \int_a^x g(t)dt \tag{8}$$

and denote

$$B_1(x; \alpha, \beta) := F^\alpha(x) G^\beta(x);$$

$$B_2(x; \alpha, \beta, s) := \left(\int_x^b f(t) G^{\frac{\beta - s}{\alpha}}(t)dt \right)^\alpha G^s(x);$$

$$B_3(x; \alpha, \beta, s) := \left(\int_a^x g(t) F^{\frac{\alpha - s}{\beta}}(t)dt \right)^\beta F^s(x);$$

$$B_4(x; \alpha, \beta, s) := \left(\int_a^x f(t) G^{\frac{\beta + s}{\alpha}}(t)dt \right)^\alpha G^{-s}(x);$$

$$B_5(x; \alpha, \beta, s) := \left(\int_x^b g(t) F^{\frac{\alpha + s}{\beta}}(t)dt \right)^\beta F^{-s}(x);$$

$$B_6(x; \alpha, \beta, s) := \left(\int_x^b f(t) G^{\frac{\beta}{\alpha + s}}(t)dt \right)^{\alpha + s} F^{-s}(x);$$

$$B_7(x; \alpha, \beta, s) := \left(\int_a^x g(t) F^{\frac{\alpha}{\beta + s}}(t)dt \right)^{\beta + s} G^{-s}(x);$$

$$B_8(x;\alpha,\beta,s):=\left(\int_a^x f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x), \alpha>s; \tag{9}$$

$$B_9(x;\alpha,\beta,s):=\left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x), \alpha<s;$$

$$B_{10}(x;\alpha,\beta,s):=\left(\int_x^b g(t)F^{\frac{\alpha}{\beta-s}}(t)dt\right)^{\beta-s} G^s(x), \beta>s;$$

$$B_{11}(x;\alpha,\beta,s):=\left(\int_a^x g(t)F^{\frac{\alpha}{\beta-s}}(t)dt\right)^{\beta-s} G^s(x), \beta<s;$$

$$B_{12}(x;\alpha,\beta,s,h):=\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\alpha} (h(x)+G(x))^s, \beta<s;$$

$$B_{13}(x;\alpha,\beta,s,h):=\left(\int_a^x g(t)h^{\frac{\alpha-s}{\beta}}(t)dt\right)^{\beta} (h(x)+F(x))^s, \alpha<s;$$

$$B_{14}(x;\alpha,\beta,s,h):=\left(\int_a^x f(t)\big(h(t)+G(t)\big)^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\alpha} h^{-s}(x);$$

$$B_{15}(x;\alpha,\beta,s,h):=\left(\int_x^b f(t)\big(h(t)+F(t)\big)^{\frac{\alpha+s}{\beta}}(t)dt\right)^{\beta} h^{-s}(x).$$

Then the numbers

$$B_1 \coloneqq \sup_{a<x<b} B_1(x;\alpha,\beta),\ B_i \coloneqq \sup_{a<x<b} B_i(x;\alpha,\beta,s)\ ,$$

$$i=2,3,\dots,11$$

and

$B_i \coloneqq \inf_{h\geq 0} \sup_{a<x<b} B_i(x;\alpha,\beta,s,h)$, $i=12,\ 13, 14, 15$ are equivalent with each other. The constants in the equivalence relations can depend on α,β and s.

Remark 2.5. The proof of this theorem is carried out by deriving positive constants c_i and d_i so that

$$c_i \sup_{a<x<b} B_i(x;\alpha,\beta,s) \leq \sup_{a<x<b} B_1(x;\alpha,\beta) \leq d_i \sup_{a<x<b} B_i(x;\alpha,\beta,s)\,.$$

This information is useful e.g. for obtaining better estimates for the best constant in (4). Let us go back to the Hardy inequality (7). For the case $1<q<p<\infty$ we have several equivalent conditions, e.g. the so-called Maz'ya-Rozin condition

$$B_{MR} \coloneqq \left(\int_0^\infty U^{\frac{r}{p}}(x)V^{\frac{r}{p'}}(x)w(x)dx\right)^{1/r} < \infty$$

where $U(x)=\int_x^\infty w(t)dt$ and $V(x)=\int_0^x z^{1-p'}(t)dt$ and equivalent conditions have been found by Tomasseli and by Persson and Stepanov (see Kufner *et al.* and Kufner and Persson (2003)). Finally, in Persson *et al.* (2007) these conditions have been extended to the following conditions in the following theorem. This theorem will be used to present our second result.

Theorema 2.6. Let $0<q<p<\infty, 1<p<\infty$ and $q\neq 1$. Then the Hardy inequality (7) holds for some finite constant $C_{p,q}>0$ if and only if any of the constants

$$B_{MR}(s):=\left(\int_0^\infty\left(\int_x^\infty w(t)V^{q\left(\frac{1}{p'}-s\right)}(t)dt\right)^{\frac{r}{q}}V^{rs-1}(x)z^{1-p'}(x)dx\right)^{\frac{1}{r}}<\infty, \tag{10}$$

$$B_{PS}(s):=\left(\int_0^\infty\left(\int_0^x w(t)V^{q\left(\frac{1}{p'}+s\right)}(t)dt\right)^{\frac{r}{q}}V^{-rs-1}(x)z^{1-p'}(x)dx\right)^{\frac{1}{r}}<\infty \tag{11}$$

is finite for some $s>0$. Moreover, for the best constant $C_{p,q}$ in (7) we have $C_{p,q}\approx B_{MR}(s)\approx B_{PS}(s)$.

3. CONTINUITY OF THE FOURIER OPRERATOR

In this section we present our main results in the cases $\mathrm{p}\leq \mathrm{q}$ and $\mathrm{q}<p$. The first provides the sufficient conditions for the Fourier transform weighted norm inequality (4) in the case $1<p\leq \mathrm{q}<\infty$.

The case $1<p\leq q<\infty$**.**

Theorem 3.1. Let $1<p\leq \mathrm{q}<\infty$, $0<s<1$ and u,v,h are positive weight functions on $\mathbb{R}^n$. Denote

$$U(x)=\int_x^\infty \frac{u^*(1/t)}{t^2}\,dt \quad \text{and} \quad V(x)=\int_0^x\,[(1/v)^*(t)]^{p'-1}dt$$

where u^*, $\left(\frac{1}{v}\right)^*$ are the decreasing rearrangements of $u,\frac{1}{v}$, respectively.

Define

$$A_1(x;s):=\left(\int_x^\infty \frac{u^*(1/t)}{t^2}\ V^{q\left(\frac{1}{p'}-s\right)}(t)dt\right)^{1/q} V^s(x);$$

$$A_2(x;s):=\left(\int_0^x [(1/v)^*(t)]^{p'-1} U^{p'\left(\frac{1}{q}-s\right)}(t)dt\right)^{1/p'} U^s(x);$$

$$A_3(x;s):=\left(\int_0^x \frac{u^*(1/t)}{t^2}\ V^{q\left(\frac{1}{p'}+s\right)}(t)dt\right)^{1/q} V^{-s}(x);$$

$$A_4(x;s):=\left(\int_x^\infty [(1/v)^*(t)]^{p'-1} U^{p'\left(\frac{1}{q}+s\right)}(t)dt\right)^{1/p'} U^{-s}(x);$$

$$A_5(x;s):=\left(\int_x^\infty \frac{u^*(1/t)}{t^2}\ V^{\frac{q}{p'(1+sq)}}(t)dt\right)^{\frac{1+sq}{q}} U^{-s}(x);$$

$$A_6(x;s):=\left(\int_0^x [(1/v)^*(t)]^{p'-1} U^{\frac{p'}{q(1+sp')}}(t)dt\right)^{\frac{1+sp'}{p'}} V^{-s}(x);$$

$$A_7(x;s):=\left(\int_0^x \frac{u^*(1/t)}{t^2}\ V^{\frac{q}{p'(1-sq)}}(t)dt\right)^{\frac{1-sq}{q}} U^s(x), qs<1;$$

$$A_8(x;s):=\left(\int_x^\infty \frac{u^*(1/t)}{t^2}\ V^{\frac{q}{p'(1-sq)}}(t)dt\right)^{\frac{1-sq}{q}} U^s(x), qs>1;$$

$$A_9(x;s):=\left(\int_x^\infty [(1/v)^*(t)]^{p'-1} U^{\frac{p'}{q(1-sp')}}(t)dt\right)^{\frac{1-sp'}{p'}} V^s(x), p's<1;$$

$$A_{10}(x;s):=\left(\int_0^x [(1/v)^*(t)]^{p'-1} U^{\frac{p'}{q(1-sp')}}(t)dt\right)^{\frac{1-sp'}{p'}} V^s(x),\ p's>1;$$

$$A_{11}(x;s,h):=\left(\int_x^\infty \frac{u^*(1/t)}{t^2}\ h(t)^{\mathrm{q}\left(\frac{1}{\mathrm{p}'}-\mathrm{s}\right)} dt\right)^{\frac{1}{q}} (h(x)+V(x))^s,\ p's>1;$$

$$A_{13}(x;s,h):=\left(\int_0^x [(1/v)^*(t)]^{p'-1} h(t)^{\mathrm{p}'\left(\frac{1}{\mathrm{q}}-\mathrm{s}\right)} dt\right)^{\frac{1}{p'}} (h(x)+U(x))^s,$$

$qs>1;$

$$A_{14}(x;s,h):=\left(\int_0^x \frac{u^*(1/t)}{t^2}\ \left(h(t)+V(t)\right)^{\mathrm{q}\left(\frac{1}{\mathrm{p}'}+\mathrm{s}\right)} dt\right)^{\frac{1}{q}} h^{-s}(x);$$

$$A_{15}(x;s,h):=\left(\int_x^\infty \frac{u^*(1/t)}{t^2}\ \left(h(t)+U(t)\right)^{\mathrm{p}'\left(\frac{1}{\mathrm{q}}+\mathrm{s}\right)} dt\right)^{\frac{1}{p'}} h^{-s}(x).$$

Then the Fourier inequality (4) holds for all non-negative measurable function $f \in L_v^p(\mathbb{R}^n)$ if any of the quantities

$$A_i(s) \coloneqq sup_{x>0}\, A_i(x;s), i = 1,2,\dots,10$$

and

$$A_i(s) \coloneqq \inf_{h>0} sup_{x>0}\, A_i(x;s,h), i = 11,\dots,14$$

is finite.

Proof. To prove the theorem we first show the equivalences of the quantities A_i to A from (5) by using Theorem 2.4 and then we use Theorem 2.3. In (8) and (9) we put $a = 0, b = \infty$, $f(x) = \frac{u^*(1/x)}{x^2}$, $g(x) = [(1/v)^*(x)]^{p'-1}$ so that $F(x) = U(x)$, $G(x) = V(x)$ and choose $\alpha = 1/q$, $\beta = 1/p'$. Then the assertion follows from the fact that the following expressions:

$$A_1(s) \coloneqq sup_{x>0}\, B_2\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_2(s) \coloneqq sup_{x>0}\, B_3\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_3(s) \coloneqq sup_{x>0}\, B_4\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_4(s) \coloneqq sup_{x>0}\, B_5\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_5(s) \coloneqq sup_{x>0}\, B_6\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_6(s) \coloneqq sup_{x>0}\, B_7\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_7(s) \coloneqq sup_{x>0}\, B_8\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_8(s) \coloneqq sup_{x>0}\, B_9\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_9(s) \coloneqq sup_{x>0}\, B_{10}\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_{10}(s) \coloneqq sup_{x>0}\, B_{11}\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_{11}(h;s) \coloneqq \inf_{h>0} sup_{x>0}\, B_{12}\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_{12}(h;s) \coloneqq \inf_{h>0} sup_{x>0}\, B_{13}\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_{13}(h;s) \coloneqq \inf_{h>0} sup_{x>0}\, B_{14}\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0;$$

$$A_{14}(h;s) \coloneqq \inf_{h>0} sup_{x>0}\, B_{15}\left(x;\frac{1}{q},\frac{1}{p'},s\right),\ \ s > 0$$

are all equivalent to A from (5) according to Theorem 2.4 and the finiteness of A, because of Theorem 2.3, is sufficient for the inequality (4) to hold. Theorem is proved.

The case $1 < q < p < \infty$. In this part, from the same point of view we study the case $1 < q < p < \infty$.

Denote $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. In this case it is known that (4) holds for some constant $C > 0$, if any of the following quantities is finite (see Benedetto and Heinig (2003))

$$B_1 := \left(\int_0^\infty U^{r/q}(x)V^{r/q'}(x)[(1/v)^*(x)]^{p'-1}dx\right)^{1/r} < \infty \tag{12}$$

or

$$B_2 := \left(\int_0^\infty \left(\int_0^x \frac{u^*(1/t)}{t^2} V^q(t)dt\right)^{r/q} V^{-r/q}(x)[(1/v)^*(x)]^{p'-1}dx\right)^{1/r} < \infty \tag{13}$$

where $U(x) = \int_x^\infty \frac{u^*(1/t)}{t^2}\, dt$ and $V(x) = \int_0^x [(1/v)^*(t)]^{p'-1}dt.$

If we integrate the quantities B_1 and B_2 by parts, then we can obtain the following quantities, satisfying Hardy's inequality and fitting the conditions of Maz'ya-Rozin and Persson-Stepanov:

$$\mathfrak{B}_1 := \left(\int_0^\infty U^{\frac{r}{q}}(x)V^{\frac{r}{q'}}(x)\frac{u^*(1/x)}{x^2}\, dx\right)^{\frac{1}{r}} < \infty, \tag{14}$$

$$\mathfrak{B}_2 := \left(\int_0^\infty \left(\int_0^x \frac{u^*(1/t)}{t^2} V^q(t)\, dt\right)^{r/p} V^{q-r/p}(x)\frac{u^*(1/x)}{x^2}\, dx\right)^{1/r} < \infty. \tag{15}$$

Moreover, we have the following relation between (12) and (14), $B_1^r = \frac{p'}{q}\mathfrak{B}_1^r$. The relation between (13) and (15) reads as $B_2^r = \frac{p}{q}\mathfrak{B}_2^r$ if $V(\infty) = \infty$ and $B_2^r = \frac{p}{q}\mathfrak{B}_2^r - \frac{p}{r}\left(\int_0^x \frac{u^*(1/t)}{t^2}V^q(t)\, dt\right)^{r/p} V^{q-r/p}(x)$ if $0 < V(\infty) < \infty$. We apply conditions (10) and (11) with $w(t)$ and z(t) replaced by $\frac{u^*(1/t)}{t^2}$ and $[(1/v)^*(t)]^{-1}$ respectively, and we get the following functions:

$B_1(s) :=$

$$\left(\int_0^\infty \left(\int_x^\infty \frac{u^*(1/t)}{t^2}\, V^{q(\frac{1}{p'}-s)}(t)dt\right)^{r/q} V^{sr-1}(x)[(1/v)^*(x)]^{p'-1}dx\right)^{1/r}, \tag{16}$$

$B_2(s) :=$

$$\left(\int_0^\infty \left(\int_0^x \frac{u^*\left(\frac{1}{t}\right)}{t^2}\, V^{q\left(\frac{1}{p'}+s\right)}(t)dt\right)^{\frac{r}{q}} V^{-sr-1}(x)[(1/v)^*(x)]^{p'-1}dx\right)^{\frac{1}{r}}. \tag{17}$$

Using in (16) and (17) integration by parts, we have

$\mathfrak{B}_1(s)$

$$:= \left(\int_0^\infty \left(\int_x^\infty \frac{u^*(1/t)}{t^2}\, V^{q(\frac{1}{p'}-s)}(t)dt\right)^{r/q} V^{q(\frac{1}{p'}-s)+sr}(x)\frac{u^*(1/x)}{x^2}dx\right)^{1/r},$$

$\mathfrak{B}_2(s)$

$$:= \left(\int_0^\infty \left(\int_0^x \frac{u^*(1/t)}{t^2}\, V^{q(\frac{1}{p'}+s)}(t)dt\right)^{r/q} V^{q\left(\frac{1}{p'}+s\right)-sr}(x)\frac{u^*(1/x)}{x^2}dx\right)^{1/r}.$$

Then we have the following relations $B_1^r(s) = \frac{1}{sq}\mathfrak{B}_1^r(s)$ and $B_2^r(s) = \frac{1}{sq}\mathfrak{B}_2^r(s)$ if $V(\infty) = \infty$,
$B_2^r(s) = \frac{1}{sq}\mathfrak{B}_2^r(s) - \frac{p}{r}\left(\int_0^x \frac{u^*(1/t)}{t^2} V^q(t)\, dt\right)^{r/p} V^{q-r/p}(\infty)$ if $0 < V(\infty) < \infty$.

Now we can formulate our second main result.

Theorem 3.2. Let $1 < q < \mathrm{p} < \infty$. Then the Fourier inequality (2.1) holds for some finite constant $\mathrm{C} > 0$ if any of the constants $\mathrm{B_i(s)}$ or $\mathfrak{B}_\mathrm{i}(\mathrm{s})$ $i = 1, 2$ is finite for some s > 0.

Proof. The result follows from the fact that the (equivalent) conditions for $\mathrm{B_{MR}(s)}$, $\mathrm{B_{PS}(s)}$ coincide with the conditions for $\mathrm{B_1(s)}$, $\mathrm{B_2(s)}$ respectively, if we replace $w(t)$, $z(t)$ by $\frac{u^*(1/t)}{t^2}$, $[(1/v)^*(t)]^{-1}$ respectively, then we get proof of the theorem.

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