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On Cyclotomic Polynomial Coefficients

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ABSTRACT

For a positive integer $n \geq 1$ the n-th cyclotomic polynomial is defined by $\Phi_n(z) = \prod_{\zeta^n=1}(z-\zeta)$, where ζ are the primitive n-th roots of unity. These polynomials are known to possess many interesting properties. In this article we establish an integral formula for the coefficients of the cyclotomic polynomial, we then discuss the direct and alternate sums of coefficients, as well as the mid-term of $\Phi_n(z)$. Finally, these results are used in the computation of certain trigonometric integrals.

Keywords: Euler totient function, cyclotomic polynomial, coefficients of cyclotomic polynomials, integral formula, direct sum of coefficients, mid-term coefficient, alternate sum of coefficients.

1. Introduction

Recall that the n-th cyclotomic polynomial Φ_n is defined by

$$\Phi_n(z) = \prod_{\substack{1 \le k \le n-1 \\ \gcd(k,n)=1}} (z - \zeta_n^k), \tag{1}$$

where $\zeta_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ denotes the first primitive root of order n of the unity. It is well known that the degree of Φ_n is $\varphi(n)$, where φ denotes Euler's totient function.

The first six cyclotomic polynomials are

$$\Phi_1(z) = z - 1,\ \Phi_2(z) = z + 1,\ \Phi_3(z) = z^2 + z + 1,$$
$$\Phi_4(z) = z^2 + 1,\ \Phi_5(z) = z^4 + z^3 + z^2 + z + 1,\ \Phi_6(z) = z^2 - z + 1.$$

Using the Möbius function μ, defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k; \\ 0 & \text{if } n = p^2 m, \end{cases}$$

where p, p_1, $\ldots$, p_k are primes, an alternative form of (1) is obtained by the multiplicative version of the Möbius inversion formula, as

$$\Phi_n(z) = \prod_{d|n} \left(x^d - 1\right)^{\mu(n/d)}.$$

The following formula is also known to hold,

$$z^n - 1 = \prod_{d|n} \Phi_d(z). \tag{2}$$

For $m \ge 1$, one obtains by induction from (2), that

$$\Phi_{2^m}(z) = z^{2^{m-1}} + 1. \tag{3}$$

Also, it is well-known that every cyclotomic polynomial has integer coefficients and is irreducible over $\mathbb{Z}$ (Ireland and Rosen, 1990, Theorem 1, p.195).

The cyclotomic polynomials have many important applications. Here we only mention that they are used in the proof of the following classical results:

1.(Gauss-Wanzel) It is possible to construct the regular n-gon with a straightedge and compass if and only if n has the form $2^k p_1 p_2 \cdots p_r$, where $k \geq 0$ and the p_j's are distinct Fermat primes.

2.(Dirichlet). Let n be a positive integer. Then there exist infinitely many prime numbers p with $p \equiv 1 \pmod{n}$.

3.(Wedderburn). Any finite associative skew field R is commutative, i.e., it is a field.

Numerous interesting properties of the cyclotomic polynomials and their coefficients have been discovered over more than a hundred years. First, the polynomials up to $n < 105$ only have 0, 1 and -1 as coefficients. In 1883, Mignotti pointed out that -2 first appears as the coefficient of z^7 of Φ_{105}, while Φ_n only has the coefficients 0 and ± 1, whenever n is a product of at most two distinct primes. Then, 2 first appears for $n = 165$, while all coefficients of Φ_n do not exceed 2 in absolute value for $n < 385$. Later, in 1895 Bang showed that for $n = pqr$ with $p < q < r$ odd primes, no coefficient of Φ_n is larger than $p - 1$. An important breakthrough came in 1931, when Schur showed that the coefficients of cyclotomic polynomials can be arbitrarily large in absolute value. The history of these early results can be found in Lehmer (1936). Later, Suzuki (1987) proved that any integer number can be a coefficient of a cyclotomic polynomial of a certain degree. For more historical details regarding the study of cyclotomic polynomials and their coefficients we refer the reader to Erdös (1946), Erdös and Vaughan (1974), Ji and Li (2008), and to the monograph Bachman (1993).

From the extensive list of references devoted to the study of the coefficients of cyclotomic polynomials, we here mention the papers Bateman (1949), Bateman et al. (1981), Endo (1975), Maier (1990), Maier (1993), Maier (1995) or Vaughan (1975). Other important results concerning the coefficients of cyclotomic polynomials and their properties have also featured in the works Beiter (1964), Beiter (1968), Beiter (1971), Carlitz (1966), Dresden (2004) and Lam and Leung (1996).

Attempts at computing explicit formulae for the coefficients (at least in theory) are mentioned in (Sándor and Crstici, 2004, p.258-259), using fairly complicated expressions. Using Stirling and Bernoulli numbers, Lehmer (1966) has obtained formulae for the coefficients of $\Phi_n(z+1)$ as polynomials with rational coefficients of certain Jordan functions. In Grytczuk and Tropak (1991), the authors provided a numerical method for the determination of the cyclotomic polynomial coefficients.

Integer sequences related to the coefficients of cyclotomic polynomials can be found in the OEIS (2018).

In this article we first give a unitary integral formula for all the coefficients of Φ_n (Theorem 2.1). We then present some applications, related to the direct and alternate sums of coefficients, and to the mid-term of Φ_n, in connection with some trigonometric integrals (Theorems 3.1 and 3.2).

2. An integral formula for the coefficients of Φ_n

The proof of the main result uses the following known identity involving Euler's totient function.

Lemma 2.1. *Let $n \geq 3$ be a positive integer. The following formula holds:*

$$\sum_{\substack{1\leq k\leq n-1\\ \gcd(k,n)=1}} k = \frac{n}{2}\varphi(n). \tag{4}$$

Proof. If $k \in \{1, \ldots, n-1\}$ is relatively prime with n, then so is number $n-k$. There are $\varphi(n)$ numbers relatively prime with n in total, hence there are $\varphi(n)/2$ pairs of numbers which sum up to n. □

Writing the polynomial $\Phi_n(z)$ in algebraic form, we obtain:

$$\Phi_n(z) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} z^j,$$

where $c_j^{(n)}$, $j = 0, 1, \ldots, \varphi(n)$, are the coefficients of $\Phi_n(z)$. As we have already mentioned, all the coefficients $c_j^{(n)}$ are integers.

In order to get a unitary formula for $c_j^{(n)}$, we introduce the function

$$\Lambda_n(t) = \prod_{\substack{1\leq k\leq n-1\\ \gcd(k,n)=1}} \sin\left(t - \frac{k\pi}{n}\right). \tag{5}$$

For $n = 1, 2$ one obtains the following expressions,

$$\Lambda_1(t) = \sin(t-\pi) = -\sin t,$$
$$\Lambda_2(t) = \sin\left(t - \frac{\pi}{2}\right) = -\cos t.$$

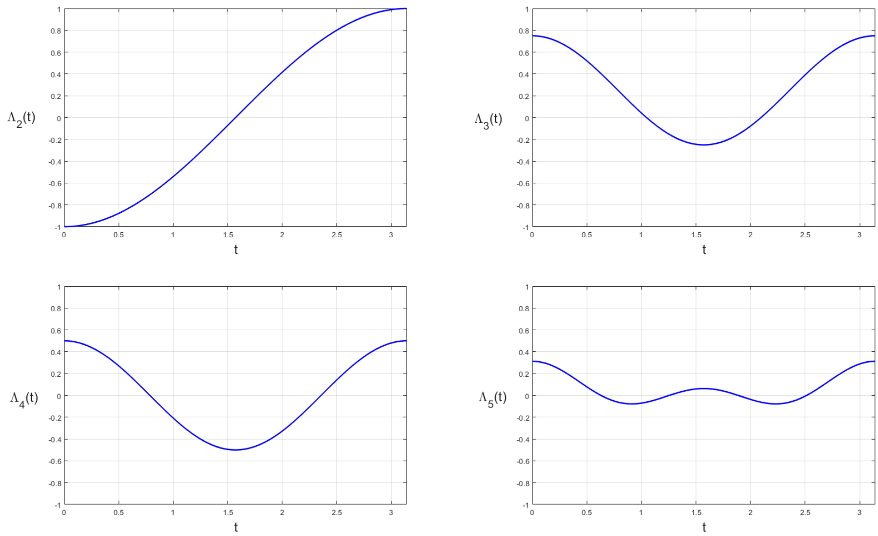

Figure 1: Function $\Lambda_n(t)$ evaluated for $0 \leq t \leq \pi$ and (a) $n = 2$; (b) $n = 3$; (c) $n = 4$; (d) $n = 5$.

As polynomials $\Phi_1(z)$ and $\Phi_2(z)$ are linear, in what follows we assume $n \geq 3$.

Theorem 2.1. *The coefficients $c_j^{(n)}$ are given by the following integral formula:*

$$c_j^{(n)} = \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t) \cdot \cos(\varphi(n) - 2j)t \, \mathrm{d}t, \quad j = 0, 1, \ldots, \varphi(n). \tag{6}$$

Proof. Denote by $\zeta_n = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ and let $z = \cos 2t + i \sin 2t$ for $t \in [0, 2\pi]$. By the well-known de Moivre formula and work with complex numbers in polar form (see for example Andreescu and Andrica (2014)), for $k = 1, \ldots, n$, we have

$$\begin{aligned} z - \zeta_n^k &= \left(\cos 2t - \cos \frac{2k\pi}{n}\right) + i \left(\sin 2t - \sin \frac{2k\pi}{n}\right) \\ &= -2 \sin\left(t - \frac{k\pi}{n}\right) \sin\left(t + \frac{k\pi}{n}\right) + 2i \sin\left(t - \frac{k\pi}{n}\right) \cos\left(t + \frac{k\pi}{n}\right) \\ &= 2i \sin\left(t - \frac{k\pi}{n}\right) \left[\cos\left(t + \frac{k\pi}{n}\right) + i \sin\left(t + \frac{k\pi}{n}\right)\right]. \end{aligned}$$

By Lemma 2.1, for $n \geq 3$, one can write the polynomial $\Phi_n(z)$ in the form

$$\begin{aligned}
\Phi_n(z) &= \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} (z-\zeta_n^k) \\
&= (2i)^{\varphi(n)} \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \sin\left(t-\frac{k\pi}{n}\right)\left[\cos\left(t+\frac{k\pi}{n}\right)+i\sin\left(t+\frac{k\pi}{n}\right)\right] \\
&= (2i)^{\varphi(n)}\Lambda_n(t)\cdot \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \left[\cos\left(t+\frac{k\pi}{n}\right)+i\sin\left(t+\frac{k\pi}{n}\right)\right] \\
&= (2i)^{\varphi(n)}\Lambda_n(t)\cdot\left[\cos\left(\varphi(n)t+\frac{\varphi(n)\pi}{2}\right)+i\sin\left(\varphi(n)t+\frac{\varphi(n)\pi}{2}\right)\right] \\
&= (2i)^{\varphi(n)}(-1)^{\frac{\varphi(n)}{2}}\Lambda_n(t)\cdot[\cos\varphi(n)t+i\sin\varphi(n)t] \\
&= 2^{\varphi(n)}\Lambda_n(t)\cdot[\cos\varphi(n)t+i\sin\varphi(n)t],
\end{aligned}$$

where we have used that $\varphi(n)$ is even for $n \geq 3$, and the multiplication of complex numbers in polar form. For every $j = 0, 1, \ldots, \varphi(n)$, one may write

$$\begin{aligned}
c_j^{(n)} + \sum_{k\neq j} c_k^{(n)} z^{k-j} &= z^{-j} \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} (z-\zeta_n^k) \\
&= 2^{\varphi(n)}\Lambda_n(t)\cdot(\cos 2jt - i\sin 2jt)\left[\cos\varphi(n)t+i\sin\varphi(n)t\right] \\
&= 2^{\varphi(n)}\Lambda_n(t)\cdot[\cos(\varphi(n)-2j)t+i\sin(\varphi(n)-2j)t].
\end{aligned}$$

Integrating on the interval $[0, \pi]$ we obtain formula (6). This is true since the integral of z^{k-j} over $[0, \pi]$ vanishes whenever $k \neq j$. $\square$

In addition, from the proof of the integral formula (6) it follows that

$$\int_0^{\pi} \Lambda_n(t)\cdot\sin(\varphi(n)-2j)t\,\mathrm{d}t = 0, \quad j = 0, 1, \ldots, \varphi(n). \tag{7}$$

The coefficients of the cyclotomic polynomial are known to be reciprocal. Here we give an elegant proof based on the integral formula (6).

Theorem 2.2. *The cyclotomic polynomial $\Phi_n(z)$ is reciprocal, that is its coefficients satisfy the following symmetry relations*

$$c_j^{(n)} = c_{\varphi(n)-j}^{(n)}, \quad j = 0, 1, \ldots, \varphi(n).$$

Proof. Using formula (6), for every $j = 0, 1, \ldots, \varphi(n)$, we have

$$\begin{aligned} c^{(n)}_{\varphi(n)-j} &= \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t) \cdot \cos\left(\varphi(n) - 2(\varphi(n) - j)\right) t \, \mathrm{d}t \\ &= \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t) \cdot \cos\left(2j - \varphi(n)\right) t \, \mathrm{d}t \\ &= \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t) \cdot \cos\left(\varphi(n) - 2j\right) t \, \mathrm{d}t = c^{(n)}_j. \end{aligned}$$

□

Here, we provide some numerical examples which illustrate the result in Theorem 2.1 when n is a product of three or more distinct odd primes. For the calculations in this section we have used Matlab® and formula (6).

Example 1. $n = 105 = 3 \times 5 \times 7$**.** This is the first time when the cyclotomic polynomial has a coefficient that is not equal to $0, 1, -1$, discovered by Mignotti in 1883, as we have already mentioned in the introduction. The degree of the polynomial is $\varphi(105) = 48$ and $c^{(105)}_7 = c^{(105)}_{41} = -2$.

Example 2. $n = 165 = 3 \times 5 \times 11$**.** This is the first time when the cyclotomic polynomial has 2 as a coefficient. The degree of the polynomial is $\varphi(165) = 80$ and we have $c^{(165)}_{16} = c^{(165)}_{17} = c^{(165)}_{31} = c^{(165)}_{32} = c^{(165)}_{33} = c^{(165)}_{47} = c^{(165)}_{48} = c^{(165)}_{49} = c^{(165)}_{63} = c^{(165)}_{64} = 2$, while -2 does not feature as a coefficient.

Example 3. $n = 385 = 5 \times 7 \times 11$**.** This is the first time when the cyclotomic polynomial has -3 as a coefficient. In this case the degree of the polynomial is $\varphi(385) = 240$ and we have $c^{(385)}_{119} = c^{(385)}_{120} = c^{(385)}_{121} = -3$, while 3 does not feature in the coefficients list.

Example 4. $n = 1155 = 3 \times 5 \times 7 \times 11$**.** This is the first product of four primes. The cyclotomic polynomial has all the numbers between -3 and 3 as a coefficients. The degree of the polynomial is $\varphi(1155) = 480$. In this case we have the values $c^{(1155)}_j = 3$ for $j = 95, 115, 117, 146, 229, 240, 251, 334, 363, 365, 385$, and $c^{(1155)}_j = -3$ for $j = 94, 116, 194, 286, 364, 386$.

One can easily check that the conclusion of Theorem 2.2 is verified in all these numerical examples, as for instance we have $c^{(1155)}_{95} = c^{(1155)}_{385}$, while clearly, $95 + 385 = 480 = \varphi(1155)$.

3. Applications

In this section we discuss integral formulae for the direct and alternate sums of coefficients and for the mid-term of the cyclotomic polynomial.

3.1 Direct sum of coefficients $\Phi_n(1)$

This produces the following integer sequence

$$0, 2, 3, 2, 5, 1, 7, 2, 3, 1, 11, 1, 13, 1, 1, 2, 17, 1, 19, 1, 1, 1, 23, \dots,$$

indexed under the labels A020500 and A014963 in OEIS (2018). Interestingly, this sequence has stretches of 1 of arbitrary length.

The following explicit formula is known for the sum of coefficients:

$$\Phi_n(1) = \begin{cases} 0 & \text{if } n = 1; \\ p & \text{if } n = p^m; \\ 1 & \text{otherwise}, \end{cases} \tag{8}$$

where $m \geq 1$ is an integer and p is prime. We give an integral equivalent.

Theorem 3.1. *Let $n \geq 3$. Expression $\Phi_n(1)$ has the following integral formula:*

$$\Phi_n(1) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} = \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cdot \frac{\sin(\varphi(n)+1)t}{\sin t} \, \mathrm{d}t. \tag{9}$$

Proof. By the formula (6) for the coefficients $c_j^{(n)}$ one obtains

$$\begin{aligned} \Phi_n(1) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} &= \sum_{j=0}^{\varphi(n)} \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cdot \cos(\varphi(n) - 2j)t \, \mathrm{d}t, \\ &= \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cdot \left[\sum_{j=0}^{\varphi(n)} \cos(\varphi(n) - 2j)t \right] \mathrm{d}t. \end{aligned}$$

The sum of cosines can be evaluated as follows,

$$\begin{aligned} \sum_{j=0}^{\varphi(n)} \cos(\varphi(n) - 2j)t \cdot \sin t &= \sum_{j=0}^{\varphi(n)} \frac{1}{2} \left[\sin(\varphi(n) - 2j + 1)t - \sin(\varphi(n) - 2j - 1)t \right] \\ &= \frac{1}{2} \left[\sin(\varphi(n) + 1)t - \sin(-\varphi(n) - 1)t \right] \\ &= \sin(\varphi(n) + 1)t. \end{aligned}$$

Finally, dividing by $\sin t$ one obtains

$$\sum_{j=0}^{\varphi(n)} \cos(\varphi(n)-2j)t = \frac{\sin(\varphi(n)+1)t}{\sin t}, \tag{10}$$

which leads to the final formula (9). □

Combining formula (9) with (8) we obtain:

Corollary 1. *For every integer $n \geq 3$, the following formula holds*

$$\int_0^{\pi} \Lambda_n(t) \cdot \frac{\sin(\varphi(n)+1)t}{\sin t}\, \mathrm{d}t = \begin{cases} \frac{p\pi}{2^{\varphi(n)}} & \text{if } n = p^m; \\ \frac{\pi}{2^{\varphi(n)}} & \text{otherwise,} \end{cases}$$

where p is a prime number.

Remark 3.1. *The righ-hand side of* (10) *is in fact a polynomial in* $\sin t$ *and* $\cos t$*, hence the integral in formula* (9) *is not improper. Because $\varphi(n)$ is even, by de Moivre's formula one can write*

$$\begin{aligned} \cos(\varphi(n)+1)t + i\sin(\varphi(n)+1)t &= (\cos t + i \sin t)^{\varphi(n)+1} \\ &= \sum_{k=0}^{\varphi(n)+1} i^k \binom{\varphi(n)+1}{k} (\cos t)^{\varphi(n)+1-k} (\sin t)^k . \end{aligned}$$

Separating the real and imaginary parts, the following identities are obtained

$$\cos(\varphi(n)+1)t = \sum_{j=0}^{\varphi(n)/2} \left[(-1)^j \binom{\varphi(n)+1}{2j} (\cos t)^{\varphi(n)-2j} (\sin t)^{2j}\right] \cos t,$$

$$\sin(\varphi(n)+1)t = \sum_{j=0}^{\varphi(n)/2} \left[(-1)^j \binom{\varphi(n)+1}{2j+1} (\cos t)^{\varphi(n)-2j} (\sin t)^{2j}\right] \sin t.$$

3.2 The mid-term of $\Phi_n(z)$

The middle coefficients for $n \geq 3$ produce the following integer sequence

$$1, 0, 1, -1, 1, 0, 1, 1, 1, -1, 1, -1, -1, 0, 1, -1, 1, 1, \dots,$$

indexed as A094754 in OEIS. The terms are also given by the integral formula

$$m_n := c^{(n)}_{\frac{\varphi(n)}{2}} = \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t)\, \mathrm{d}t. \tag{11}$$

Because the polynomial $\Phi_n(z)$ is reciprocal we have $\Phi_n(1) = 2a + m_n$, for some integer a. By formula (8), $\Phi_n(1)$ is an odd number if n is not a power of 2. As a result, m_n has to be an odd number in this case. Moreover, we have $m_n = 0$ if and only if $n = 2^m$ for some $m \geq 2$. By this remark and formula (11) we obtain the following property of the function $\Lambda_n(t)$:

$$\int_0^{\pi} \Lambda_n(t)\,\mathrm{d}t = 0, \quad \text{if and only if } n = 2^m, \quad m \geq 1.$$

While the terms of the sequence m_n seem to be equal to -1, 0 or 1, other negative and positive values appear. For example $m_{385} = -3$, $m_{6545} = -5$ and $m_{7735} = -7$, while $m_{1155} = 3$, $m_{4785} = 5$, and $m_{11305} = 19$. Some of these values are mentioned in the paper by Dresden (2004). This suggests that the following property may hold:

Conjecture. *Every odd integer can be the mid-coefficient of some cyclotomic polynomial.*

3.3 Alternate sum of coefficients $\Phi_n(-1)$

The following explicit formula is known for the alternate sum of coefficients.

$$\Phi_n(-1) = \begin{cases} -2 & \text{if } n = 1; \\ 0 & \text{if } n = 2; \\ p & \text{if } n = 2p^m; \\ 1 & \text{otherwise,} \end{cases} \tag{12}$$

where $m \geq 1$ is an integer. We sketch the proof of the above formula when n is odd. By formula (2) one obtains:

$$\frac{z^n - 1}{z - 1} = \prod_{d|n,\, d>1} \Phi_d(z), \tag{13}$$

which evaluated for an odd number n at $z = -1$ gives

$$1 = \prod_{d|n,\, d>1} \Phi_d(-1). \tag{14}$$

Clearly, this indicates that $\Phi_n(-1) = 1$ whenever n is prime. Then, all divisors d of n are odd, and one can use an inductive argument to show that $\Phi_n(-1) = 1$.

As an integer sequence, the terms of $\Phi_n(-1)$ recover entry A020513 in OEIS:

$$-2, 0, 1, 2, 1, 3, 1, 2, 1, 5, 1, 1, 1, 7, 1, 2, 1, 3, 1, 1, 1, 11, 1, 1, 1, 13, \dots.$$

By the formula (12), one can prove that this sequence also has stretches of 1 of arbitrary length, which is not mentioned in the OEIS.

We shall explore an integral equivalent of this result.

Theorem 3.2. *Let $n \geq 3$. Terms $\Phi_n(-1)$ have the following integral formula:*

$$\Phi_n(-1) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} (-1)^j = \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cdot \frac{\cos(\varphi(n)+1)t}{\cos t} \, \mathrm{d}t. \tag{15}$$

Proof. By the formula (6) for the coefficients $c_j^{(n)}$ one obtains

$$\begin{aligned}\Phi_n(-1) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} (-1)^j &= \sum_{j=0}^{\varphi(n)} \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t)(-1)^j \cos(\varphi(n)-2j)t \, \mathrm{d}t, \\ &= \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cdot \left[\sum_{j=0}^{\varphi(n)} (-1)^j \cos(\varphi(n)-2j)t \right] \mathrm{d}t.\end{aligned}$$

For a fixed $j \in \{0, \dots, \varphi(n)\}$, the following identity holds

$$(-1)^j \cos(\varphi(n)-2j)t = \cos\left[(\varphi(n)-2j)t - j\pi\right] = \cos\left[\varphi(n)t - 2j\left(t + \frac{\pi}{2}\right)\right].$$

Multiplying by $\sin\left(t + \frac{\pi}{2}\right)$ one obtains

$$\begin{aligned}&2\cos\left[\varphi(n)t - 2j\left(t+\frac{\pi}{2}\right)\right] \cdot \sin\left(t+\frac{\pi}{2}\right) = \\ &\sin\left[\varphi(n)t - (2j-1)\left(t+\frac{\pi}{2}\right)\right] - \sin\left[\varphi(n)t - (2j+1)\left(t+\frac{\pi}{2}\right)\right].\end{aligned}$$

Summing for $j = 0, \dots, \varphi(n)$, we obtain the telescopic sum

$$\begin{aligned}&\sum_{j=0}^{\varphi(n)} \cos(\varphi(n)-2j)t \cdot (-1)^j \cdot \sin\left(t+\frac{\pi}{2}\right) \\ &= \frac{\sin\left[(\varphi(n)+1)t + \frac{\pi}{2}\right] - \sin\left[-(\varphi(n)+1)t - \frac{\pi}{2} - \varphi(n)\pi\right]}{2} \\ &= \frac{\sin\left[(\varphi(n)+1)t + \frac{\pi}{2}\right] + \sin\left[(\varphi(n)+1)t + \frac{\pi}{2} + \varphi(n)\pi\right]}{2} \\ &= \frac{\cos\left[(\varphi(n)+1)t\right] + \cos\left[(\varphi(n)+1)t + \varphi(n)\pi\right]}{2}.\end{aligned}$$

By the identities $\sin(x+\pi/2)=\cos x$ and $\cos(x+k\pi)=(-1)^k\cos x,\ k\in\mathbb{Z}$, one obtains the following formula

$$\sum_{j=0}^{\varphi(n)}(-1)^j\cos(\varphi(n)-2j)t=\frac{\cos(\varphi(n)+1)t}{\cos t}\cdot\frac{1+(-1)^{\varphi(n)}}{2}.$$

Since $\varphi(n)$ is even for $n\geq 3$, the above formula gives

$$\sum_{j=0}^{\varphi(n)}(-1)^j\cos(\varphi(n)-2j)t=\frac{\cos(\varphi(n)+1)t}{\cos t},$$

which leads to the final formula (15). □

The formulae (12) and (15) have the following consequence:

Corollary 2. *If $n\geq 3$, then*

$$\int_0^\pi \Lambda_n(t)\cdot\frac{\cos(\varphi(n)+1)t}{\cos t}\,\mathrm{d}t=\begin{cases}\frac{p\pi}{2^{\varphi(n)}} & \text{if } n=2p^m;\\ \frac{\pi}{2^{\varphi(n)}} & \text{otherwise.}\end{cases}$$

where p is a prime number.

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