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I-localized Sequence in Two Normed Spaces

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ABSTRACT

In this paper, our aim is to introduce the $\mathcal{I}$-localized and the $\mathcal{I}^*$-localized sequences in 2-normed spaces and study the fundamental properties of $\mathcal{I}$-localized sequences. Under some conditions, we show that $\mathcal{I}$-localized sequence is an $\mathcal{I}$-Cauchy sequence. Furthermore, the definition of uniformly $\mathcal{I}$-localized sequences in 2-normed spaces is given, and some results related with this concept are obtained.

Keywords: Ideal convergence, 2-normed space, ideal localor of the sequence, Cauchy sequence, uniformly localized sequence.

1. Introduction and Preliminaries

If X is a metric space with a metric $d(\cdot,\cdot)$ and (x_n) is a sequence of points in X, the sequence (x_n) is said to be localized in some subset $M \subset X$ if the number sequence $\alpha_n = d(x_n, x)$ converges for all $x \in M$. This definition that can be thought as a generalizaton of Cauchy sequence in metric space was introduced by Krivonosov (1974). The author also obtained important results related with the closure operators in metric spaces. Recently, Nabiev et al. (2019) have extended the concepts and results, which were given by Krivonosov (1974), by changing the usual limit to the statistical limit in metric spaces. Also, Nabiev et al. (2020) have generalized the notion of ideal localized sequence using the concept of ideal of subset of the set $\mathbb{N}$ of positive integers and obtained important results.

To present our results we need to make some definitons and notations (see Kostyrko et al. (2000); Kostyrko et al. (2005)).

Recall that the family $\mathcal{I} \subset 2^X$ for a non-empty set X is called an ideal if and only if $P \cup R \in \mathcal{I}$ for every $P, R \in \mathcal{I}$, and $R \in \mathcal{I}$ for every $P \in \mathcal{I}$ and $R \subset P$. A non-empty family of sets $\mathcal{F} \subset 2^X$ is a filter on X if and only if $\emptyset \notin \mathcal{F}$, $P \cap R \in \mathcal{F}$ for every $P, R \in \mathcal{F}$, and $R \in \mathcal{F}$ for every $P \in \mathcal{F}$ and every $R \supset P$. An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. The $\mathcal{I} \subset 2^X$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \backslash P : P \in \mathcal{I}\}$ is a filter on X. A non-trivial ideal $\mathcal{I} \subset 2^X$ is called admissible if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ said to hold the property (AP) if for every family $\{P_n\}_{n\in\mathbb{N}}$ with $P_n \cap P_k = \emptyset\ (n \neq k)$, $P_n \in \mathcal{I}\ (n \in \mathbb{N})$ there is a family $\{R_n\}_{n\in\mathbb{N}}$ such that $(P_k \backslash R_n) \cup (R_k \backslash P_k)$ for all $k \in \mathbb{N}$ and a limit set $R = \bigcup\limits_{k=1}^{\infty} R_k \in \mathcal{I}$ (Kostyrko et al. (2000)).

Definition 1.1. *(Gähler (1993)) Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent; (ii) $\|x, y\| = \|y, x\|$; (iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$; (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. The pair $(X, \|.,.\|)$ is then called a 2-normed space.*

Let $a, b \in X$ and let us define for each $\varepsilon > 0$ the ε-neighborhood of the points a, b as the set

$$U_\varepsilon(a, b) = \{c : \|a - c, b - c\| < \varepsilon\}.$$

As it is known (Raymond and Cho (2001)) that the family of all sets

$$W_{\Sigma} = \bigcap_{i=1}^{n} U_{\varepsilon_i}(a_i, b_i)$$

with arbitrary pairs $\Sigma = \{(b_1, \varepsilon_1), ..., (b_n, \varepsilon_n)\}$ forms a complete system of neighborhoods of the point $a \in X$. Note that a set M in a linear 2-normed space $(X, \|.,.\|)$ is said to be bounded if $\beta(M) < \infty$, where

$$\beta(M) = \sup\{\|a - c, b - c\| : a, b, c \in M\}.$$

We also suppose that for any $\varepsilon > 0$ there exists a neighborhood U of O such that for all points a^* and b^* $\|a^*, b^*\| < \varepsilon$.

Now, we recall some definitions in Gürdal and Açı k (2008), Şahiner et al. (2007).

Definition 1.2. *(Şahiner et al. (2007)) A sequence $(x_n)_{n\in\mathbb{N}}$ in 2-normed space $(X, \|.,.\|)$ is said to be $\mathcal{I}$-convergence to $\mu \in X$ if and only if*

$$A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - \mu, z\| \geq \varepsilon\} \in \mathcal{I}$$

for any $\varepsilon > 0$ and every nonzero $z \in X$. it will be represented by $\mathcal{I}\text{-}\lim_{n\to\infty} \|x_n, z\| = \mu$.

Definition 1.3. *(Şahiner et al. (2007)) A sequence $(x_n)_{n\in\mathbb{N}}$ in 2-normed space $(X, \|.,.\|)$ is said to be $\mathcal{I}$-Cauchy sequence if and only if there exists $n_0 \in \mathbb{N}$ such that*

$$A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x_{n_0}, z\| \geq \varepsilon\} \in \mathcal{I}$$

for each $\varepsilon > 0$ and every nonzero $z \in X$.

Definition 1.4. *(Gürdal and Açı k (2008)) A sequence $(x_n)_{n\in\mathbb{N}}$ in 2-normed space $(X, \|.,.\|)$ is said to be $\mathcal{I}^*$-convergent to $\mu \in X$ if and only if there exists a set $B \in \mathcal{F}(\mathcal{I})$ such that $\lim_{k\to\infty} \|x_{m_k} - \mu, z\| = 0$ and $B = \{b_1 < b_2 < ... < b_k < ...\} \subset \mathbb{N}$.*

Definition 1.5. *(Gürdal and Açı k (2008)) A sequence $(x_n)_{n\in\mathbb{N}}$ in 2-normed space $(X, \|.,.\|)$ is said to be $\mathcal{I}^*$-Cauchy sequence if and only if there is a set $B = \{b_1 < b_2 < ... < b_k\}$ such that*

$$\lim_{k,p\to\infty} \left\|x_{m_k} - x_{m_p}, z\right\| = 0.$$

It is well-known that $\mathcal{I}^*$-Cauchy sequences and $\mathcal{I}^*$-convergent mean $\mathcal{I}$-Cauchy sequences and $\mathcal{I}$-convergent, respectively. Furthermore, if $\mathcal{I}$ is an ideal satisfying the property (AP), then $\mathcal{I}$ and $\mathcal{I}^*$-convergent coincide (see Gürdal and Açı k (2008)). So, in this case, $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy sequences are the same (see citega). More property and fact about ideal convergence and 2-normed space are contained, for instance, in Dündar and Altay (2014), Gürdal and Yamancı (2015), Mohiuddine and Aiyub (2012), Mohiuddine et al. (2012, 2010), Mursaleen and Alotaibi (2011), Mursaleen et al. (2010), Mursaleen and Mohiuddine (2010, 2012), Yamancıand Gürdal (2014), Yamancıet al. (2020).

In this paper, our purpose is to introduce the $\mathcal{I}$-localized and the $\mathcal{I}^*$-localized sequences in 2-normed spaces and research the fundamental properties of $\mathcal{I}$-localized sequences. Under some conditions, we show that $\mathcal{I}$-localized sequence is an $\mathcal{I}$-Cauchy sequence. Furthermore, the definition of uniformly $\mathcal{I}$-localized sequences in 2-normed spaces is given, and some results related with this concept are obtained.

2. $\mathcal{I}$ and $\mathcal{I}^*$-localized sequences in 2-normed spaces

In this part, we give some new definitions and notations.

Definition 2.1. (a) *A sequence $(x_n)_{n\in\mathbb{N}}$ in 2-normed space $(X, \|.,.\|)$ is called as the $\mathcal{I}$-localized in the subset $M \subset X$ if and only if $\mathcal{I}\text{-}\lim_{n\to\infty} \|x_n - x, z\|$ exists for every $x, z \in M$, that is, the real number sequence $\|x_n - x, z\|$ is $\mathcal{I}$-convergent.*

(b) *the maximal set on which a sequence (x_n) is $\mathcal{I}$-localized is said to be $\mathcal{I}$-localor of (x_n) and it is denoted by $loc_{\mathcal{I}}(x_n)$.*

(c) *A sequence (x_n) in 2-normed space $(X, \|.,.\|)$ is said to be $\mathcal{I}$-localized everywhere if (x_n) is $\mathcal{I}$-localor of (x_n) coincides with X.*

(d) *A sequence (x_n) in 2-normed space $(X, \|.,.\|)$ is called as the $\mathcal{I}$-localized in itself if*

$$\{n \in \mathbb{N} : x_n \notin loc_{\mathcal{I}}(x_n)\} \subset \mathcal{I}.$$

We are able to easily see from above definition that if (x_n) is an $\mathcal{I}$-Cauchy sequence, then so is $\mathcal{I}$-localized everywhere. Actually, owing to

$$|\|x_n - x, z\| - \|x_{n_0} - x, z\|| \leqslant \|x_n - x_{n_0}, z\|$$

we have

$$\{n \in \mathbb{N} : |\|x_n - x, z\| - \|x_{n_0} - x, z\|| \geqslant \varepsilon\} \subset \{n \in \mathbb{N} : \|x_n - x_{n_0}, z\| \geqslant \varepsilon\}.$$

So, the sequence is $\mathcal{I}$-localized if it is $\mathcal{I}$-Cauchy sequence.

Also, we are able to say that each $\mathcal{I}$-convergence sequence is $\mathcal{I}$-localized. Note that if $\mathcal{I}$ is an admissible ideal, then every localized sequence in 2-normed space $(X, \|.,.\|)$ is $\mathcal{I}$-localized sequence in $(X, \|.,.\|)$.

Definition 2.2. *We say the sequence (x_n) to be $\mathcal{I}^*$-localized in 2-normed space $(X, \|.,.\|)$ if and only if the number sequence $\|x_n - x, z\|$ is $\mathcal{I}^*$-convergent for every $x, z \in X$.*

From above definition, we are able to say that every $\mathcal{I}^*$-Cauchy sequence or $\mathcal{I}^*$-convergent in 2-normed space $(X, \|.,.\|)$ is $\mathcal{I}^*$-localized in $(X, \|.,.\|)$.

Note that for admissible ideal, $\mathcal{I}^*$-convergence and $\mathcal{I}^*$-Cauchy criteria imply $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy criteria, respectively. Furthermore, for the admissible ideal satisfying the property (AP) the notions $\mathcal{I}$-convergent and $\mathcal{I}^*$-convergent; $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy sequences coincide (Gürdal (2006)).

Lemma 2.1. *Let X be a 2-linear normed space and $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. If a sequence $(x_n) \subset X$ is $\mathcal{I}^*$-localized on the set $M \subset X$, then (x_n) is $\mathcal{I}$-localized on the set M and $loc_{\mathcal{I}_*}(x_n) \subset loc_{\mathcal{I}}(x_n)$.*

Proof. Assume that (x_n) is $\mathcal{I}^*$-localized on M. Then, there is a set $P \in \mathcal{I}$ such that

$$\lim_{j\to\infty} \|x_j - x, z\|$$

exists for each $x, z \in M$ and $P^C = \mathbb{N}\backslash P = \{p_1 < p_2 < ... < p_j\}$. Then, the sequence $\|x_n - x, z\|$ is an $\mathcal{I}^*$-Cauchy sequence, which means that $\|x_n - x, z\|$ is a $\mathcal{I}$-Cauchy sequences. Therefore, the number sequence $\|x_n - x, z\|$ is $\mathcal{I}$-convergent, which gives that (x_n) is $\mathcal{I}$-localized on the set M. $\square$

Lemma 2.2. *Assume $(X, \|.,.\|)$ is a 2-normed space. There are two cases if X has limit point or not.*

(i) If X has no limit point, then $\mathcal{I}$ and $\mathcal{I}^*$-localized sequences are the same in X and $\mathcal{I}_{loc}(x_n) = \mathcal{I}^*_{loc}(x_n)$ for any $(x_n) \in X$.

(ii) If X has a limit point ψ, then there is an admissible ideal $\mathcal{I}$ for which there exists an $\mathcal{I}$-localized sequence $(y_n) \subset X$ such that (y_n) is not $\mathcal{I}^*$-localized.

Proof. (i) Let X has no any limit point. Then, the notions $\mathcal{I}$ and $\mathcal{I}^*$-convergence coincide in X (see Kostyrko et al. (2000)). Hence, if (x_n) is $\mathcal{I}$-localized, then it is also $\mathcal{I}^*$-localized, and from the Lemma 2.1 we get $\mathcal{I}_{loc}(x_n) = \mathcal{I}^*_{loc}(x_n)$.

(ii) Assume that ψ be a limit point of X. Then, there is a sequence (x_n) such that $\lim_{n\to\infty} \|x_n - \psi, z\| = 0$. Let $K = \bigcup_{j=1}^{\infty} K_j$, $K_j = \left\{\alpha^{j-1}(2s-1) : s \in \mathbb{N}\right\} (j = 1, 2, ...)$ is a decomposition of integers and Γ is an ideal of sets $P \subset \mathbb{N}$ such that every P intersects only a finite element of K_j. Let us define the sequence $y_n = x_j$ for $n \in K_j$. Then, the sequence (y_n) is $\mathcal{I}$-localized in X. So, $\mathcal{I}$-$\lim_{n\to\infty} \|y_n - x, z\| = \beta(x)$ for every $x, z \in X$.

Let us take $\|x_n - x, z\| = \beta_n(x)$. It is clear to prove that $\mathcal{I}$-$\lim_{n\to\infty} \|x_n - x, z\| = \beta(x)$. If we have $\mathcal{I}^*$-$\lim_{n\to\infty} \|x_n - x, z\| = \beta(x)$ for some $x, z \in X$, then there is a $A \in \Gamma$ such that

$$\lim_{k\to\infty} \|x_{m_k} - x, z\| = \beta(x)$$

for $B = \{b_1 < b_2 < ... < b_k < ...\} = \mathbb{N}\backslash A$. In terms of the definition of Γ, we are able to get integer $\ell \in \mathbb{N}$ such that $A \subset \Delta_1 \cup ... \cup \Delta_\ell$. But then, $\Delta_{\ell+1} \subset \mathbb{N}\backslash A = B$. Hence, we get $\|x_{m_k} - x, z\| \to \|x_{\ell+1} - x, z\|$ for infinity many k's which contradicts that $\|x_{m_k} - x, z\| \to \beta(x)$. So, the sequence (y_n) is $\mathcal{I}$-localized but it is not $\mathcal{I}^*$-localized, which completes the proof. $\square$

We are able to deduce that if (x_n) is $\mathcal{I}$-localized sequence and $\mathcal{I}$ is an admissible ideal satisfying the property (AP), then $\mathcal{I}^*$-$\lim_{m\to\infty} \|x_m - x, z\|$ exists for every x, z. So, $\mathcal{I}$-localized sequence for any admissible ideal with the property (AP) is $\mathcal{I}^*$-localized. On the contrary, if the 2-normed space $(X, \|.,.\|)$ has at least one limit point, then $\mathcal{I}$ has the property (AP).

3. Basic properties of ideal localized sequences

In this section, our purpose is to research other properties of $\mathcal{I}$-localized sequences.

Proposition 3.1. *Let (x_n) be an $\mathcal{I}$-localized sequence in a linear 2-normed space $(X, \|.,.\|)$. Then (x_n) is $\mathcal{I}$-bounded.*

Proof. Suppose that (x_n) is $\mathcal{I}$-localized. Then, the number sequence $\|x_n - x, z\|$ is $\mathcal{I}$-convergent for some $x, z \in X$. This means that $\{n \in \mathbb{N} : \|x_n - x, z\| > K\} \in \mathcal{I}$ for some $K > 0$. As a result, the sequence (x_n) is $\mathcal{I}$-bounded. $\square$

Proposition 3.2. *Let $\mathcal{I}$ be an admissible ideal satisfying the property (AP) and $M = loc_{\mathcal{I}}(x_n)$. Also, a point $y \in X$ be such that there exists $x \in M$ for* *any* $\varepsilon > 0$ *and every nonzero* $z \in M$ *such that*

$$\{n \in \mathbb{N} : |\|x - x_n, z\| - \|y - x_n, z\|| \geqslant \varepsilon\} \in \mathcal{I}. \tag{1}$$

Then $y \in M$.

Proof. To prove that the number sequence $\|x_n - y, z\|$ is an $\mathcal{I}$-Cauchy sequence is enough. Let be $\varepsilon > 0$ and $x \in M = loc_{\mathcal{I}}(x_n)$ is a point satisfying the property (1). Owing to the (AP) property of $\mathcal{I}$, we get

$$|\|x - x_{k_n}, z\| - \|y - x_{k_n}, z\|| \to 0$$

and

$$|\|x_{k_n} - x, z\| - \|x_{k_m} - x, z\|| \to 0$$

as $m, n \to \infty$, where $K = \{k_1 < k_2 < ... < k_n < ...\} \in \mathcal{F}(\mathcal{I})$. Therefore, there is $n_0 \in \mathbb{N}$ for any $\varepsilon > 0$ and every nonzero $z \in M$ such that

$$|\|x - x_{k_n}, z\| - \|y - x_{k_n}, z\|| < \frac{\varepsilon}{3} \tag{2}$$

$$|\|x - x_{k_n}, z\| - \|x - x_{k_m}, z\|| < \frac{\varepsilon}{3}. \tag{3}$$

for all $n \geq n_0$, $m \geq m_0$.

Using (2) and (3) together with the following

$$\begin{aligned}
&|\|y - x_{k_n}, z\| - \|y - x_{k_m}, z\||\\
&\leq |\|y - x_{k_n}, z\| - \|x - x_{k_n}, z\|| + |\|x - x_{k_n}, z\| - \|x - x_{k_m}, z\|| +\\
&+ |\|x - x_{k_m}, z\| - \|y - x_{k_n}, z\||
\end{aligned}$$

we obtain

$$|\|y - x_{k_n}, z\| - \|y - x_{k_m}, z\|| < \varepsilon$$

for all $n \geq n_0$, $m \geq n_0$. So,

$$|\|y - x_{k_n}, z\| - \|y - x_{k_m}, z\|| \to 0 \text{ as } m, n \to \infty$$

for the $K = (k_n) \subset N$ and $K \in \mathcal{F}(\mathcal{I})$. Hence $\|y - x_n, z\|$ is an $\mathcal{I}$-Cauchy sequence, which completes the proof. $\square$

Definition 3.1. *A point* a *in a* 2*-normed space* $(X, \|., .\|)$ *is called a limit point of a set* M *in* X *if for an arbitrary* $\Sigma = \{(b_1, \varepsilon_1), ..., (b_n, \varepsilon_n)\}$, *there is a point* $a_\Sigma \in M$, $a_\Sigma \neq a$ *such that* $a_\Sigma \in W_\Sigma(a)$.

Moreover, a subset $Y \subset X$ is called a closed subset of X if Y contains every its limit point. If Y^0 is the set of all points of a subset $Y \subset X$, then the set $\overline{Y} = Y \cup Y^0$ is called the closure of the set Y.

Proposition 3.3. *$\mathcal{I}$-localor of any sequence is a closed subset of the 2-normed space $(X, \|.,.\|)$.*

Proof. Let $y \in \overline{loc_{\mathcal{I}}}\,(x_n)$. Then, there is a point $x \in loc_{\mathcal{I}}\,(x_n)$ for arbitrary $\Sigma = \{(b_1, \varepsilon_1), ..., (b_n, \varepsilon_n)\}$ such that $x \neq y$ and $x \in W_\Sigma(y)$. Thus, for any $\varepsilon > 0$ and every $z \in loc_{\mathcal{I}}\,(x_n)$

$$\{n \in \mathbb{N} : |\|x - x_n, z\| - \|y - x_n, z\|| \geqslant \varepsilon\} \in \mathcal{I}$$

owing to

$$|\|x - x_n, z\| - \|y - x_n, z\|| \leq \|y - x_n, z\| < \varepsilon \text{ for every } n \in \mathbb{N}.$$

In conclusion, the hypothesis of Proposition 3.2 is satisfied. and then we reach that $y \in loc_{\mathcal{I}}\,(x_n)$, that is, $loc_{\mathcal{I}}\,(x_n)$ is closed. $\square$

Recall that the point y is an $\mathcal{I}$-limit point of the sequence (x_n) in 2-normed space $(X, \|.,.\|)$ if there is a set $K = \{k_1 < k_2 < ... < k_n\} \subset \mathbb{N}$ such that $K \notin \mathcal{I}$ and $\lim_{n\to\infty} \|x_{k_n} - y, z\| = 0$. A point ξ is said to be an $\mathcal{I}$-cluster point of the sequence (x_n) if for each $\varepsilon > 0$ and every $z \in X$

$$\{n \in \mathbb{N} : \|x_n - \xi, z\| < \varepsilon\} \notin \mathcal{I}$$

(Gürdal (2006)).

We can give the following proposition owing to $|\|x_n - y, z\| - \|x - y, z\|| \leq \|x_n - x, z\|$.

Proposition 3.4. *Let $y \in X$ be an $\mathcal{I}$-limit point (an $\mathcal{I}$-cluster point) of a sequence (x_n) in 2-normed space $(X, \|.,.\|)$. Then, the number $\|y - x, z\|$ is an $\mathcal{I}$-limit point (an $\mathcal{I}$-cluster point) of the sequence $\{\|x_n - x, z\|\}$ for each $x \in X$ and every nonzero $z \in X$.*

Proposition 3.5. *Let (x_n) be an $\mathcal{I}$-localized sequence in 2-normed space $(X, \|.,.\|)$. Then, all $\mathcal{I}$-limit points ($\mathcal{I}$-cluster points) of the sequnce have the same distance from every point x of the locator $loc_{\mathcal{I}}\,(x_n)$.*

Proof. Let y_1 and y_2 be two $\mathcal{I}$-limit points of the sequence (x_n) in 2-normed space $(X, \|.,.\|)$. Then, the numbers $\|y_1 - x, z\|$ and $\|y_2 - x, z\|$ are $\mathcal{I}$-limit points of the $\mathcal{I}$-convergent sequence $\|x - x_n, z\|$. Eventually, $\|y_1 - x, z\| = \|y_2 - x, z\|$. $\square$

Proposition 3.6. *$loc_{\mathcal{I}}(x_n)$ contain at most one $\mathcal{I}$-limit ($\mathcal{I}$-cluster) point of the sequence (x_n) in 2-normed space $(X, \|.,.\|)$. Especially, everywhere localized sequence has at most one $\mathcal{I}$-limit ($\mathcal{I}$-cluster) point.*

Proof. Let $x, y \in loc_{\mathcal{I}}(x_n)$ be two $\mathcal{I}$-limit points of the sequence (x_n). then by the Proposition 3.5 $\|x - x, z\| = \|x - y, z\|$. But $\|x - x, z\| = 0$. This implies $\|x - y, z\| = 0$ for $x \neq y$, which is a contradiction. □

Definition 3.2. *Let (x_n) be the $\mathcal{I}$-localized sequence (x_n) with the $\mathcal{I}$-localor $M = loc_{\mathcal{I}}(x_n)$. The number*

$$\lambda = \inf_{x \in M} \left(\mathcal{I}\text{-} \lim_{n \to \infty} \|x - x_n, z\| \right)$$

is called as the $\mathcal{I}$-barrier of (x_n).

Theorem 3.1. *Let $(X, \|.,.\|)$ be an 2-normed space and let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an ideal satisfying the (AP) property. Then, an $\mathcal{I}$-localized sequence is $\mathcal{I}$-Cauchy sequence if and only if $\lambda = 0$.*

Proof. Assume that (x_n) is an $\mathcal{I}$-Cauchy sequence in 2-normed space $(X, \|.,.\|)$. Then, there is a set $R = \{r_1 < r_2 < ... < r_n\} \subset \mathbb{N}$ such that $R \in \mathcal{F}(\mathcal{I})$ and $\lim_{n,m \to \infty} \|x_{r_n} - x_{r_m}, z\| = 0$. As a consequence, there is a $n_0 \in \mathbb{N}$ for every $\varepsilon > 0$ and every $z \in X$ such that

$$\left\| x_{r_n} - x_{r_{n_0}}, z \right\| < \varepsilon$$

for all $n \geq n_0$. Because (x_n) is $\mathcal{I}$-localized sequence, $\mathcal{I}\text{-}\lim_{n \to \infty} \left\| x_n - x_{r_{n_0}}, z \right\|$ exists and we get

$$\mathcal{I} - \lim_{n \to \infty} \left\| x_n - x_{r_{n_0}}, z \right\| \leq \varepsilon.$$

Therefore, $\lambda \leq \varepsilon$. Because $\varepsilon > 0$ is arbitrary, we obtain $\lambda = 0$.

Now assume the $\lambda = 0$. Then, there is a $x \in loc_{\mathcal{I}}(x_n)$ for every $\varepsilon > 0$ and every nonzero $z \in X$ such that

$$\|x, z\| = \mathcal{I} - \lim_{n \to \infty} \|x - x_n, z\| < \frac{\varepsilon}{2}.$$

Then

$$\left\{ n \in \mathbb{N} : |\|x, z\| - \|x - x_n, z\|| \geq \frac{\varepsilon}{2} - \|x, z\| \right\} \in \mathcal{I}.$$

Hence, we have

$$\left\{ n \in \mathbb{N} : \|x - x_n, z\| \geq \frac{\varepsilon}{2} \right\} \in \mathcal{I}.$$

So, $\mathcal{I}$-$\lim_{n\to\infty} \|x - x_n, z\| = 0$, which gives us that (x_n) is $\mathcal{I}$-Cauchy sequence. $\square$

Remark 3.1. *As is seen in the proof of Theorem 1, if $\lambda = 0$ then an ideal $\mathcal{I}$ do not need to have (AP) properties. In other saying, if $\mathcal{I}$-barrier of a localized sequence is equal to zero, then it is an $\mathcal{I}$-Cauchy sequence itself.*

Theorem 3.2. *Let the sequence (x_n) be an $\mathcal{I}$-localized in itself and (x_n) contains an $\mathcal{I}$-nonthin Cauchy subsequence. Then, (x_n) is an $\mathcal{I}$-Cauchy sequence itself.*

Proof. Suppose that (y_n) is an $\mathcal{I}$-nonthin Cauchy subsequence. We are able to assume that $loc_{\mathcal{I}}(x_n)$ contains the all elements of (y_n). We see that

$$\inf_{y_n} \lim_{m\to\infty} \|y_m - y_n, z\| = 0$$

because (y_n) is a Cauchy sequence by Theorem 3.1. Also, because (x_n) is $\mathcal{I}$-localized in itself

$$\mathcal{I}\text{-}\lim_{m\to\infty} \|x_m - y_n, z\| = \mathcal{I}\text{-}\lim_{m\to\infty} \|y_m - y_n, z\| = 0.$$

Then, we get that the $\mathcal{I}$-barrier of (x_n) is equal to zero. Consequently, from the Remark 3.1 we can say that the (x_n) is $\mathcal{I}$-Cauchy sequence. $\square$

Let $a \in X$, $\delta > 0$ and $\mathcal{I} \subset 2^{\mathbb{N}}$ is an admissible ideal. Recall that the sequence (x_n) in 2-normed space $(X, \|.,.\|)$ is called $\mathcal{I}$-bounded if there is a subset $K = \{k_1 < k_2 < ... < k_n \subset ...\} \subset \mathbb{N}$ such that $K \in \mathcal{F}(\mathcal{I})$ and $\left(x_{k_n}\right) \subset U_\delta(0, z)$, where $U_\delta(0, z)$ is some neighborhood of the origin. Obviously, $\left(x_{k_n}\right)$ is a bounded sequence in X sequence and has a localized in itself subsequence. In conclusion, the following assertion is also correct.

Proposition 3.7. *Let (x_n) be an $\mathcal{I}$-bounded sequence in 2-normed space $(X, \|.,.\|)$. Then, it has an $\mathcal{I}$-localized in itself subsequence.*

Theorem 3.3. *Let (x_n) have an $\mathcal{I}$-localized in itself subsequence of $(X, \|.,.\|)$. If it is an $\mathcal{I}$-Cauchy sequence, then each bounded set in 2-normed space $(X, \|.,.\|)$ is totally bounded.*

Proof. Assume that (x_n) is an $\mathcal{I}$-localized in itself sequence of 2-normed space $(X, \|.,.\|)$, but the claim is not correct. Then, there is a bounded subset $M \subset X$ such that $\|x_n - x_m, z\| > \varepsilon$ $(n \neq m)$ for some $\varepsilon > 0$, every nonzero $z \in M$ and some sequence $(x_n) \subset M$. Because (x_n) is bounded from Proposition 7, (x_n)

has an $\mathcal{I}$-localized in itself sequence (x_n'). Due to $\|x_n' - x_m', z\| > \varepsilon$ for any $n \neq m$, the subsequence is not an Cauchy sequence, which is a contradict with our assumption. □

Definition 3.3. *An infinite subset $M \subset X$ is thick relatively to a nonempty subset $Y \subset X$ if for each $\varepsilon > 0$ there is the a point $y \in Y$ such that the neighborhood $U_\varepsilon(0, z)$ has infinitely many points of M. In particular, if the set M is thick relatively to its subset $Y \subset M$, then M is called thick in itself.*

Proposition 3.8. *If each bounded infinite set of X is thick in itself, then each $\mathcal{I}$-localized in itself sequence of X is an $\mathcal{I}$-Cauchy sequence.*

Proof. Let the supposition be correct and let (x_n) be an $\mathcal{I}$-localized in itself sequence of $(X, \|.,.\|)$. Then, (x_n) is an $\mathcal{I}$-bounded sequence in $(X, \|.,.\|)$, which means that there is an infinite set M of points of (x_n) such that M is a bounded subset of X. From the supposition, M is thick in itself. So, we can pick the $x_k \in M$ for every $\varepsilon > 0$ such that the neighborhood $U_\varepsilon(0, z)$ contains infinitely many points of $x_1', ..., x_n', ...$ of X. Hence, the sequence $(\|x_n' - x_k, z\|)$ is $\mathcal{I}$-convergent and

$$\mathcal{I} - \lim_{n \to n_0} \|x_n' - x_K, z\| \leq \varepsilon.$$

Therefore, the $\mathcal{I}$-barrier of (x_n) is equal to zero, which shows that (x_n) is an $\mathcal{I}$-Cauchy sequence. □

Definition 3.4. *A sequence (x_n) in 2-normed space $(X, \|.,.\|)$ is called as the uniformly $\mathcal{I}$-localized on a subset $M \subset X$ if the sequence $\{\|x - x_n, z\|\}$ is uniformly $\mathcal{I}$-converges for all $x, z \in M$.*

Proposition 3.9. *Let sequence (x_n) be uniformly $\mathcal{I}$-localized on the set $M \subset X$ and $w \in Y$ is such that for every $\varepsilon > 0$ and every nonzero z in M there is $y \in M$ such that*

$$\{n \in \mathbb{N} : |\|w - x_n, z\| - \|y - x_n, z\|| \geqslant \varepsilon\} \in \mathcal{I}$$

Then $w \in loc_{\mathcal{I}}(x_n)$ and (x_n) is uniformly $\mathcal{I}$-localized on the set of such points w.

Since the proof of Proposition is analog to Proposition 3.2, we omit it.

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