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Hermite-Hadamard Like Inequalities for Exponentially Subadditive Functions via Fractional Integrals

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Abstract

In this work, through the Riemann-Liouville fractional integrals, we give Hermite-Hadamard type inequalities for exponentially sub-additive functions. For the product of exponentially sub-additive functions, we present fractional integral inequalities. It is also shown that the results proved here are the refinements and extensions of several existing results in the field of Hermite-Hadamard like inequalities.

Keywords: subadditive functions; Riemann-Liouville fractional integrals; Hermite-Hadamard inequalities.

1 Introduction

C. Hermite and J. Hadamard are the founders of well-known inequality which is called Hermite-Hadamard inequality (see, e.g.[12, p.137]). In the theory of convexity, the Hermite-Hadamard inequality is well-established inequality with many applications and geometrical interpretations. This inequality states that if a function $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex, then for $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$, we have

$$\phi\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \phi(\mu) d\mu \leq \frac{\phi(\varkappa_1) + \phi(\varkappa_2)}{2}. \tag{1}$$

If the given function is concave then the above inequality holds in reversed direction. This inequality can be easily captured by using the Jensen's inequality for convex functions. For more recent findings concerning (1) reader can read [4, 7, 14]. Please refer also to [20, 21, 24].

On the other hand, the main work on the general idea of subadditive functions is given by Hille and Phillips [8]. This reference also includes a part of the work of Rosenbaum [13] on subadditive functions of several variables. Additivity, subadditivity and superadditivity are important concepts both in measure theory and in several fields of mathematics and mathematical inequalities. Especially, there are a lot of examples of additive, subadditive and superadditive functions in various areas of mathematics such as norms, square roots, error function, growth rates, differential equations and integral means. Inequalities and especially subadditive functions theory is one of the most extensively developing fields not only in theoretical and applied mathematics but also in physics and the other applied sciences. Here, we mention the results of [3, 8, 9] and the corresponding references cited therein. For other reference with the same result, please refer [10, 13].

Definition 1.1. [9] *Let $H \subseteq \mathbb{R}$, A function ϕ defined on H and with range contained in the set $\mathbb{R}^+$ (positive real numbers), is considered to be subadditive on H if, for all elements μ and ν of H such that $\mu + \nu$ is an element of H*

$$\phi(\mu + \nu) \leq \phi(\mu) + \phi(\nu).$$

If equality holds, ϕ is called additive; ϕ is considered to be superadditive, if the inequality is conversed. A mapping ϕ is considered to be convex on the (possibly infinite) interval D if, for all μ and ν in D and all δ which satisfy $0 \leq \delta \leq 1$,

$$\phi(\delta\mu + (1 - \delta)\nu) \leq \delta\phi(\mu) + (1 - \delta)\phi(\nu).$$

If this inequality is reversed, ϕ is concave on D.

Remark 1.1. *If ϕ is convex and subadditive on H and if $\phi(0) = 0$, then ϕ is additive on H.*

Definition 1.2. [9] *A function $\phi : [0, \varkappa_2] \to R$, $\varkappa_2 > 0$ is considered to be starshaped if for every $\mu \in [0, \varkappa_2]$ and $\delta \in [0, 1]$ we have $\phi(\delta\mu) \leq \delta\phi(\mu)$.*

According to above definitions, if a subadditive function $\phi : A \subset [0, \infty) \to R$ is also starshaped, then ϕ is a convex function.

In [11], Özcan gave generalization of subadditive functions and established inequalities of Hermite-Hadamrd type as follows:

Definition 1.3. *A function $\phi : I \subset [0, \infty) \to \mathbb{R}$ is considered to be exponentially subadditive, if the following inequality holds*

$$\phi(\mu + \nu) \leq \frac{\phi(\mu)}{e^{s\mu}} + \frac{\phi(\nu)}{e^{s\nu}}, \tag{2}$$

for all μ, ν in I and $s \in \mathbb{R}$. A mapping ϕ is considered to be exponentially superadditive, if inequality (2) is conversed.

Remark 1.2. *If a exponentially subadditive function $\phi : A \subset [0,\infty) \to R$ is also starshaped, then ϕ is a exponentially convex function given by Awan et al. in [2].*

Remark 1.3. *It is obvious that for $s = 0$ Definition 1.3 reduces to Definition 1.1.*

Theorem 1.1. *We assume that a continuous function $\phi : I = [0,\infty) \to \mathbb{R}$ is exponentially subadditive, $\varkappa_1, \varkappa_2 \in I^\circ$ with $\varkappa_1 < \varkappa_2$. Then, the following inequalities hold:*

$$\frac{1}{2}(\varkappa_1 + \varkappa_2) \leq \frac{1}{\varkappa_2 - \varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \frac{\phi(\mu)}{e^{s\mu}} d\mu \leq \frac{1}{\varkappa_1}\int_0^{\varkappa_1} \frac{\phi(\mu)}{e^{2s\mu}} d\mu + \frac{1}{\varkappa_2}\int_0^{\varkappa_2} \frac{\phi(\mu)}{e^{2s\mu}} d\mu.$$

Remark 1.4. *In Theorem 1.1, if we consider $s = 0$. Then, Theorem 1.1 transforms into [19, Theorem 2].*

2 Main results

In this section, we give fractional version of Hermite-Hadamard inequality and related inequalities for exponentially subadditive functions.

Theorem 2.1. *Let $\phi : I = [0,\infty) \to \mathbb{R}$ be a continuous exponentially subadditive function, $\varkappa_1, \varkappa_2 \in I^\circ$ with $\varkappa_1 < \varkappa_2$. Then, the following inequalities hold for Riemann-Liouville factional integral:*

$$\begin{aligned}\frac{1}{2}\phi(\varkappa_1 + \varkappa_2) &\leq \frac{\Gamma(\alpha+1)}{2(\varkappa_2 - \varkappa_1)^\alpha}[J^\alpha_{\varkappa_1+}(h_1(\varkappa_2) + J^\alpha_{\varkappa_2-}h_1(\varkappa_1)] \\ &\leq \frac{\alpha}{2\varkappa_1^\alpha}\int_0^{\varkappa_1}\left[\mu^{\alpha-1} + (\varkappa_1 - \mu)^{\alpha-1}\right]\frac{\phi(\mu)}{e^{s\left(2\mu + \left(\frac{\varkappa_1 - \mu}{\varkappa_1}\right)\varkappa_2\right)}}d\mu \\ &\quad + \frac{\alpha}{2\varkappa_2^\alpha}\int_0^{\varkappa_2}\left[\mu^{\alpha-1} + (\varkappa_2 - \mu)^{\alpha-1}\right]\frac{\phi(\mu)}{e^{s\left(\left(\frac{\varkappa_2 - \mu}{\varkappa_2}\right)\varkappa_1 + 2\mu\right)}}d\mu, \end{aligned} \quad (3)$$

where $h_1(\mu) = \frac{\phi(\mu)}{e^{s\mu}}$.

Proof. Since ϕ is an exponentially subadditive function, we have

$$\phi(\mu + \nu) \leq \frac{\phi(\mu)}{e^{s\mu}} + \frac{\phi(\nu)}{e^{s\nu}}. \quad (4)$$

Put $\mu = \delta\varkappa_1 + (1-\delta)\varkappa_2$ and $\nu = \delta\varkappa_2 + (1-\delta)\varkappa_1$, we have

$$\begin{aligned}\phi(\varkappa_1 + \varkappa_2) &\leq \frac{\phi(\delta\varkappa_1 + (1-\delta)\varkappa_2)}{e^{s(\delta\varkappa_1 + (1-\delta)\varkappa_2)}} + \frac{\phi((1-\delta)\varkappa_1 + \delta\varkappa_2)}{e^{s((1-\delta)\varkappa_1 + \delta\varkappa_2)}} \\ &\leq \frac{\phi(\delta\varkappa_1)}{e^{s(2\delta\varkappa_1 + (1-\delta)\varkappa_2)}} + \frac{\phi((1-\delta)\varkappa_2)}{e^{s(\delta\varkappa_1 + 2(1-\delta)\varkappa_2)}} + \frac{\phi((1-\delta)\varkappa_1)}{e^{s(2(1-\delta)\varkappa_1 + \delta\varkappa_2)}} + \frac{\phi(\delta\varkappa_2)}{e^{s((1-\delta)\varkappa_1 + 2\delta\varkappa_2)}}.\end{aligned} \quad (5)$$

Multiplying each side of (5) with $\delta^{\alpha-1}$, later integrating the resultant one respecting δ on the interval $[0,1]$, we obtain that

$$\begin{aligned}\frac{1}{\alpha}\phi(\varkappa_1 + \varkappa_2) &\leq \int_0^1 \delta^{\alpha-1}\frac{\phi(\delta\varkappa_1 + (1-\delta)\varkappa_2)}{e^{s(\delta\varkappa_1 + (1-\delta)\varkappa_2)}}d\delta + \int_0^1 \delta^{\alpha-1}\frac{\phi((1-\delta)\varkappa_1 + \delta\varkappa_2)}{e^{s((1-\delta)\varkappa_1 + \delta\varkappa_2)}}d\delta. \\ &\leq \int_0^1 \delta^{\alpha-1}\frac{\phi(\delta\varkappa_1)}{e^{s(2\delta\varkappa_1 + (1-\delta)\varkappa_2)}}d\delta + \int_0^1 \delta^{\alpha-1}\frac{\phi((1-\delta)\varkappa_2)}{e^{s(\delta\varkappa_1 + 2(1-\delta)\varkappa_2)}}d\delta \\ &\quad + \int_0^1 \delta^{\alpha-1}\frac{\phi((1-\delta)\varkappa_1)}{e^{s(2(1-\delta)\varkappa_1 + \delta\varkappa_2)}}d\delta + \int_0^1 \delta^{\alpha-1}\frac{\phi(\delta\varkappa_2)}{e^{s((1-\delta)\varkappa_1 + 2\delta\varkappa_2)}}d\delta.\end{aligned} \quad (6)$$

By using change of variables of integration, we have

$$\begin{aligned}\frac{1}{\alpha}\phi(\varkappa_1+\varkappa_2) &\leq \int_{\varkappa_1}^{\varkappa_2}(\frac{\varkappa_2-\mu}{\varkappa_2-\varkappa_1})^{\alpha-1}h_1(\mu)\frac{d\mu}{\varkappa_2-\varkappa_1}+\int_{\varkappa_1}^{\varkappa_2}(\frac{\mu-\varkappa_1}{\varkappa_2-\varkappa_1})^{\alpha-1}h_1(\mu)\frac{d\mu}{\varkappa_2-\varkappa_1} \\ &\leq \frac{1}{\varkappa_1^{\alpha}}\int_0^{\varkappa_1}\left[\mu^{\alpha-1}+(\varkappa_1-\mu)^{\alpha-1}\right]\frac{\phi(\mu)}{e^{s\left(2\mu+\left(\frac{\varkappa_1-\mu}{\varkappa_1}\right)\varkappa_2\right)}}d\mu \\ &\quad+\frac{1}{\varkappa_2^{\alpha}}\int_0^{\varkappa_2}\left[\mu^{\alpha-1}+(\varkappa_2-\mu)^{\alpha-1}\right]\frac{\phi(\mu)}{e^{s\left(\left(\frac{\varkappa_2-\mu}{\varkappa_2}\right)\varkappa_1+2\mu\right)}}d\mu.\end{aligned} \tag{7}$$

Multiplying (7) by $\frac{\alpha}{2}$, we get

$$\begin{aligned}\frac{1}{2}\phi(\varkappa_1+\varkappa_2) &\leq \frac{\Gamma(\alpha+1)}{2(\varkappa_2-\varkappa_1)^{\alpha}}[J^{\alpha}_{\varkappa_1+}h_1(\varkappa_2)+J^{\alpha}_{\varkappa_2-}h_1(\varkappa_1)] \\ &\leq \frac{\alpha}{2\varkappa_1^{\alpha}}\int_0^{\varkappa_1}\left[\mu^{\alpha-1}+(\varkappa_1-\mu)^{\alpha-1}\right]\frac{\phi(\mu)}{e^{s\left(2\mu+\left(\frac{\varkappa_1-\mu}{\varkappa_1}\right)\varkappa_2\right)}}d\mu \\ &\quad+\frac{1}{\varkappa_2^{\alpha}}\int_0^{\varkappa_2}\left[\mu^{\alpha-1}+(\varkappa_2-\mu)^{\alpha-1}\right]\frac{\phi(\mu)}{e^{s\left(\left(\frac{\varkappa_2-\mu}{\varkappa_2}\right)\varkappa_1+2\mu\right)}}d\mu,\end{aligned}$$

which completes the proof of the Theorem. □

Remark 2.1. *In Theorem 2.1, if we assume $\alpha=1$. Then, we obtain Theorem 1.1.*

Remark 2.2. *In Theorem 2.1, if we consider $\alpha=1$ and $\phi(\delta\mu)\leq\delta\phi(\mu)$. Then, we obtain [2, Theorem 1].*

Remark 2.3. *In Theorem 2.1, if we take $s=0$. Then, we get [1, Theorem 7].*

Remark 2.4. *In Theorem 2.1, if we set $s=0$ and $\phi(\delta\mu)\leq\delta\phi(\mu)$. Then, we obtain [1, Corollary 1].*

Theorem 2.2. *Let $\phi,\varphi: I=[0,\infty)\to\mathbb{R}$ be two continuous exponentially subadditive functions, $\varkappa_1,\varkappa_2\in I^{\circ}$ with $\varkappa_1<\varkappa_2$. Then, the following inequality holds for Riemann-Liouville factional integrals:*

$$\begin{aligned}&\frac{1}{2}\phi(\varkappa_1+\varkappa_2)\varphi(\varkappa_1+\varkappa_2) \\ &\leq \frac{\Gamma(\alpha+1)}{2(\varkappa_2-\varkappa_1)^{\alpha}}[J^{\alpha}_{\varkappa_1+}(h_2(\varkappa_2)+J^{\alpha}_{\varkappa_2-}h_2(\varkappa_1))]\end{aligned} \tag{8}$$

$$\begin{aligned}&+\frac{\alpha}{2e^{s(\varkappa_1+\varkappa_2)}}\left[\frac{1}{\varkappa_1^{\alpha}}\int_0^{\varkappa_1}\left[\mu^{\alpha-1}+(\varkappa_1-\mu)^{\alpha-1}\right]\frac{\phi(\mu)\varphi(\varkappa_1-\mu)}{e^{s\varkappa_1}}d\mu\right. \\ &+\frac{1}{\varkappa_2^{\alpha}}\int_0^{\varkappa_2}\left[\mu^{\alpha-1}+(\varkappa_2-\mu)^{\alpha-1}\right]\frac{\phi(\mu)\varphi(\varkappa_2-\mu)}{e^{s\varkappa_2}}d\mu \\ &+\frac{1}{\varkappa_1^{\alpha}}\int_0^{\varkappa_1}\left[\mu^{\alpha-1}+(\varkappa_1-\mu)^{\alpha-1}\right]\frac{\phi(\mu)\varphi\left(\frac{\mu\varkappa_2}{\varkappa_1}\right)}{e^{s\mu\left(\frac{\varkappa_1+\varkappa_2}{\varkappa_1}\right)}}d\mu \\ &\left.+\frac{1}{\varkappa_2^{\alpha}}\int_0^{\varkappa_2}\left[\mu^{\alpha-1}+(\varkappa_2-\mu)^{\alpha-1}\right]\frac{\phi(\mu)\varphi\left(\frac{\mu\varkappa_1}{\varkappa_2}\right)}{e^{s\mu\left(\frac{\varkappa_1+\varkappa_2}{\varkappa_2}\right)}}d\mu\right],\end{aligned}$$

where $h_2(\mu)=\frac{\phi(\mu)\varphi(\mu)}{e^{2s\mu}}$.

Proof. Since ϕ and φ are exponentially subadditive functions, so we have

$$\phi(\mu+\nu) \leq \frac{\phi(\mu)}{e^{s\mu}} + \frac{\phi(\nu)}{e^{s\nu}} \tag{9}$$

$$\varphi(\mu+\nu) \leq \frac{\varphi(\mu)}{e^{s\mu}} + \frac{\varphi(\nu)}{e^{s\nu}}. \tag{10}$$

Using $\mu = \delta\varkappa_1 + (1-\delta)\varkappa_2$ and $\nu = (1-\delta)\varkappa_1 + \delta\varkappa_2$, we have

$$\phi(\varkappa_1+\varkappa_2) \leq \frac{\phi\left(\delta\varkappa_1 + (1-\delta)\varkappa_2\right)}{e^{s(\delta\varkappa_1+(1-\delta)\varkappa_2)}} + \frac{\phi\left((1-\delta)\varkappa_1 + \delta\varkappa_2\right)}{e^{s((1-\delta)\varkappa_1+\delta\varkappa_2)}} \tag{11}$$

$$\varphi(\varkappa_1+\varkappa_2) \leq \frac{\varphi\left(\delta\varkappa_1 + (1-\delta)\varkappa_2\right)}{e^{s(\delta\varkappa_1+(1-\delta)\varkappa_2)}} + \frac{\varphi\left((1-\delta)\varkappa_1 + \delta\varkappa_2\right)}{e^{s((1-\delta)\varkappa_1+\delta\varkappa_2)}}. \tag{12}$$

From (11) and (12), we obtain

$$\begin{aligned}
&\phi(\varkappa_1+\varkappa_2)\varphi(\varkappa_1+\varkappa_2) \qquad (13)\\
\leq\; &\left[\frac{\phi\left(\delta\varkappa_1 + (1-\delta)\varkappa_2\right)}{e^{s(\delta\varkappa_1+(1-\delta)\varkappa_2)}} + \frac{\phi\left((1-\delta)\varkappa_1 + \delta\varkappa_2\right)}{e^{s((1-\delta)\varkappa_1+\delta\varkappa_2)}}\right]\\
&\times\left[\frac{\varphi\left(\delta\varkappa_1 + (1-\delta)\varkappa_2\right)}{e^{s(\delta\varkappa_1+(1-\delta)\varkappa_2)}} + \frac{\varphi\left((1-\delta)\varkappa_1 + \delta\varkappa_2\right)}{e^{s((1-\delta)\varkappa_1+\delta\varkappa_2)}}\right]\\
=\; &\frac{\phi(\delta\varkappa_1 + (1-\delta)\varkappa_2)\varphi(\delta\varkappa_1 + (1-\delta)\varkappa_2)}{e^{2s(\delta\varkappa_1+(1-\delta)\varkappa_2)}} + \frac{\phi((1-\delta)\varkappa_1 + \delta\varkappa_2)\varphi((1-\delta)\varkappa_1 + \delta\varkappa_2)}{e^{2s((1-\delta)\varkappa_1+\delta\varkappa_2)}}\\
&+\frac{\phi(\delta\varkappa_1 + (1-\delta)\varkappa_2)\varphi((1-\delta)\varkappa_1 + \delta\varkappa_2)}{e^{s(\varkappa_1+\varkappa_2)}} + \frac{\phi((1-\delta)\varkappa_1 + \delta\varkappa_2)\varphi(\delta\varkappa_1 + (1-\delta)\varkappa_2)}{e^{s(\varkappa_1+\varkappa_2)}}\\
\leq\; &\frac{\phi(\delta\varkappa_1 + (1-\delta)\varkappa_2)\varphi(\delta\varkappa_1 + (1-\delta)\varkappa_2)}{e^{2s(\delta\varkappa_1+(1-\delta)\varkappa_2)}} + \frac{\phi((1-\delta)\varkappa_1 + \delta\varkappa_2)\varphi((1-\delta)\varkappa_1 + \delta\varkappa_2)}{e^{2s((1-\delta)\varkappa_1+\delta\varkappa_2)}}\\
&+\frac{1}{e^{s(\varkappa_1+\varkappa_2)}}\left[\left(\frac{\phi\left(\delta\varkappa_1\right)}{e^{s\delta\varkappa_1}} + \frac{\phi\left(\left(1-\delta\right)\varkappa_2\right)}{e^{s((1-\delta)\varkappa_2)}}\right)\left(\frac{\varphi\left(\left(1-\delta\right)\varkappa_1\right)}{e^{s((1-\delta)\varkappa_1)}} + \frac{\varphi\left(\delta\varkappa_2\right)}{e^{s\delta\varkappa_2}}\right)\right]\\
&+\frac{1}{e^{s(\varkappa_1+\varkappa_2)}}\left[\left(\frac{\phi\left(\left(1-\delta\right)\varkappa_1\right)}{e^{s((1-\delta)\varkappa_1)}} + \frac{\phi\left(\delta\varkappa_2\right)}{e^{s\delta\varkappa_2}}\right)\left(\frac{\varphi\left(\delta\varkappa_1\right)}{e^{s\delta\varkappa_1}} + \frac{\varphi\left(\left(1-\delta\right)\varkappa_2\right)}{e^{s((1-\delta)\varkappa_2)}}\right)\right]\\
=\; &\frac{\phi(\delta\varkappa_1 + (1-\delta)\varkappa_2)\varphi(\delta\varkappa_1 + (1-\delta)\varkappa_2)}{e^{2s(\delta\varkappa_1+(1-\delta)\varkappa_2)}} + \frac{\phi((1-\delta)\varkappa_1 + \delta\varkappa_2)\varphi((1-\delta)\varkappa_1 + \delta\varkappa_2)}{e^{2s((1-\delta)\varkappa_1+\delta\varkappa_2)}}\\
&+\frac{1}{e^{s(\varkappa_1+\varkappa_2)}}\left[\frac{\phi\left(\delta\varkappa_1\right)\varphi\left(\left(1-\delta\right)\varkappa_1\right)}{e^{s\varkappa_1}} + \frac{\phi\left(\left(1-\delta\right)\varkappa_1\right)\varphi\left(\delta\varkappa_1\right)}{e^{s\varkappa_1}} + \frac{\phi\left(\delta\varkappa_2\right)\varphi\left(\left(1-\delta\right)\varkappa_2\right)}{e^{s\varkappa_2}}\right.\\
&+\frac{\phi\left(\left(1-\delta\right)\varkappa_2\right)\varphi\left(\delta\varkappa_2\right)}{e^{s\varkappa_2}} + \frac{\phi\left(\delta\varkappa_1\right)\varphi\left(\delta\varkappa_2\right)}{e^{s\delta(\varkappa_1+\varkappa_2)}} + \frac{\phi\left(\left(1-\delta\right)\varkappa_1\right)\varphi\left(\left(1-\delta\right)\varkappa_2\right)}{e^{s\delta(\varkappa_1+\varkappa_2)}}\\
&\left.+\frac{\phi\left(\delta\varkappa_2\right)\varphi\left(\delta\varkappa_1\right)}{e^{s(1-\delta)(\varkappa_1+\varkappa_2)}} + \frac{\phi\left(\left(1-\delta\right)\varkappa_2\right)\varphi\left(\left(1-\delta\right)\varkappa_1\right)}{e^{s(1-\delta)(\varkappa_1+\varkappa_2)}}\right].
\end{aligned}$$

Multiplying each side of (13) with $\delta^{\alpha-1}$, later integrating the resultant one respecting δ on the interval $[0,1]$, we get

$$
\begin{aligned}
&\frac{1}{\alpha}\phi(\varkappa_1+\varkappa_2)\varphi(\varkappa_1+\varkappa_2)\\
\leq\ & \int_0^1 \delta^{\alpha-1}\frac{\phi(\delta\varkappa_1+(1-\delta)\varkappa_2)\varphi(\delta\varkappa_1+(1-\delta)\varkappa_2)}{e^{2s(\delta\varkappa_1+(1-\delta)\varkappa_2)}}d\delta\\
&+\int_0^1 \delta^{\alpha-1}\frac{\phi((1-\delta)\varkappa_1+\delta\varkappa_2)\varphi((1-\delta)\varkappa_1+\delta\varkappa_2)}{e^{2s((1-\delta)\varkappa_1+\delta\varkappa_2)}}d\delta\\
&+\frac{1}{e^{s(\varkappa_1+\varkappa_2)}}\int_0^1 \delta^{\alpha-1}\left[\frac{\phi(\delta\varkappa_1)\varphi((1-\delta)\varkappa_1)}{e^{s\varkappa_1}}+\frac{\phi((1-\delta)\varkappa_1)\varphi(\delta\varkappa_1)}{e^{s\varkappa_1}}+\frac{\phi(\delta\varkappa_2)\varphi((1-\delta)\varkappa_2)}{e^{s\varkappa_2}}\right.\\
&+\frac{\phi((1-\delta)\varkappa_2)\varphi(\delta\varkappa_2)}{e^{s\varkappa_2}}+\frac{\phi(\delta\varkappa_1)\varphi(\delta\varkappa_2)}{e^{s\delta(\varkappa_1+\varkappa_2)}}+\frac{\phi((1-\delta)\varkappa_1)\varphi((1-\delta)\varkappa_2)}{e^{s(1-\delta)(\varkappa_1+\varkappa_2)}}\\
&\left.+\frac{\phi(\delta\varkappa_2)\varphi(\delta\varkappa_1)}{e^{s\delta(\varkappa_1+\varkappa_2)}}+\frac{\phi((1-\delta)\varkappa_2)\varphi((1-\delta)\varkappa_1)}{e^{s(1-\delta)(\varkappa_1+\varkappa_2)}}\right]d\delta.
\end{aligned}
$$

By using change of variables of integration, we obtain

$$
\begin{aligned}
&\frac{1}{\alpha}\phi(\varkappa_1+\varkappa_2)\varphi(\varkappa_1+\varkappa_2) \qquad (14)\\
\leq\ & \frac{\Gamma(\alpha)}{(\varkappa_2-\varkappa_1)^{\alpha}}[J^{\alpha}_{\varkappa_1+}(h_2(\varkappa_2)+J^{\alpha}_{\varkappa_2-}h_2(\varkappa_1)]\\
&+\frac{1}{e^{s(\varkappa_1+\varkappa_2)}}\left[\frac{1}{\varkappa_1^{\alpha}}\int_0^{\varkappa_1}\left[\mu^{\alpha-1}+(\varkappa_1-\mu)^{\alpha-1}\right]\frac{\phi(\mu)\varphi(\varkappa_1-\mu)}{e^{s\varkappa_1}}d\mu\right.\\
&+\frac{1}{\varkappa_2^{\alpha}}\int_0^{\varkappa_2}\left[\mu^{\alpha-1}+(\varkappa_2-\mu)^{\alpha-1}\right]\frac{\phi(\mu)\varphi(\varkappa_2-\mu)}{e^{s\varkappa_2}}d\mu\\
&+\frac{1}{\varkappa_1^{\alpha}}\int_0^{\varkappa_1}\left[\mu^{\alpha-1}+(\varkappa_1-\mu)^{\alpha-1}\right]\frac{\phi(\mu)\varphi\left(\frac{\mu\varkappa_2}{\varkappa_1}\right)}{e^{s\mu\left(\frac{\varkappa_1+\varkappa_2}{\varkappa_1}\right)}}d\mu\\
&\left.+\frac{1}{\varkappa_2^{\alpha}}\int_0^{\varkappa_2}\left[\mu^{\alpha-1}+(\varkappa_2-\mu)^{\alpha-1}\right]\frac{\phi(\mu)\varphi\left(\frac{\mu\varkappa_1}{\varkappa_2}\right)}{e^{s\mu\left(\frac{\varkappa_1+\varkappa_2}{\varkappa_2}\right)}}d\mu\right].
\end{aligned}
$$

Multiplying both sides of (14) by $\frac{\alpha}{2}$, we obtain the required inequality (8). □

Remark 2.5. *In Theorem 2.2, if we assume $\alpha=1$. Then, we obtain [11, Theorem 2.12].*

Remark 2.6. *In Theorem 2.2, if we consider $\alpha=1$ and $\phi(\delta\mu)\leq\delta\phi(\mu)$. Then, we acquire [2, Theorem 2].*

Remark 2.7. *If we use $s=0$ in Theorem 2.2, then we have [1, Theorem 8 (2.7)].*

Remark 2.8. *If we take $\phi(\delta\mu)\leq\delta\phi(\mu)$ and $s=0$ in Theorem 2.2, then we have [1, Corollary 2].*

Theorem 2.3. *Let $\phi,\varphi: I=[0,\infty)\to\mathbb{R}$ be two continuous exponentially subadditive functions, $\varkappa_1,\varkappa_2\in$*

I° with $\varkappa_1 < \varkappa_2$. Then we have the following inequality

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(\varkappa_2-\varkappa_1)^\alpha}\left[J^\alpha_{\varkappa_1+}\phi(\varkappa_2)\varphi(\varkappa_2)+J^\alpha_{\varkappa_2-}\phi(\varkappa_1)\varphi(\varkappa_1)\right]\\
\leq\ & \frac{\alpha}{2\varkappa_1^\alpha}\int_0^{\varkappa_1}\left[\mu^{\alpha-1}+(\varkappa_1-\mu)^{\alpha-1}\right]h_2(\mu)\,d\mu+\frac{\alpha}{2\varkappa_2^\alpha}\int_0^{\varkappa_2}\left[\mu^{\alpha-1}+(\varkappa_2-\mu)^{\alpha-1}\right]h_2(\mu)\,d\mu\\
&+\frac{\alpha}{2\varkappa_1^\alpha}\int_0^{\varkappa_1}\left[\mu^{\alpha-1}+(\varkappa_1-\mu)^{\alpha-1}\right]\frac{\phi(\mu)\varphi\left(\left(\frac{\varkappa_1-\mu}{\varkappa_1}\right)\varkappa_2\right)}{e^{s\left(\mu+\left(\frac{\varkappa_1-\mu}{\varkappa_1}\right)\varkappa_2\right)}}d\mu\\
&+\frac{\alpha}{2\varkappa_2^\alpha}\int_0^{\varkappa_2}\left[\mu^{\alpha-1}+(\varkappa_2-\mu)^{\alpha-1}\right]\frac{\phi(\mu)\varphi\left(\left(\frac{\varkappa_2-\mu}{\varkappa_2}\right)\varkappa_1\right)}{e^{s\left(\mu+\left(\frac{\varkappa_2-\mu}{\varkappa_2}\right)\varkappa_1\right)}}d\mu,
\end{aligned}$$

where $h_2(\mu)$ is same as defined in Theorem 2.2.

Proof. Since ϕ and φ are exponentially subadditive functions, we have

$$\phi(\delta\varkappa_1+(1-\delta)\varkappa_2) \leq \frac{\phi(\delta\varkappa_1)}{e^{s\delta\varkappa_1}}+\frac{\phi((1-\delta)\varkappa_1)}{e^{s(1-\delta)\varkappa_1}}, \tag{15}$$

$$\varphi(\delta\varkappa_1+(1-\delta)\varkappa_2) \leq \frac{\varphi(\delta\varkappa_1)}{e^{s\delta\varkappa_1}}+\frac{\varphi((1-\delta)\varkappa_1)}{e^{s(1-\delta)\varkappa_1}}. \tag{16}$$

From (15) and (16), we have

$$\begin{aligned}
&\phi(\delta\varkappa_1+(1-\delta)\varkappa_2)\varphi(\delta\varkappa_1+(1-\delta)\varkappa_2)\\
\leq\ & \frac{\phi(\delta\varkappa_1)\varphi(\delta\varkappa_1)}{e^{2s\delta\varkappa_1}}+\frac{\phi(\delta\varkappa_1)\varphi((1-\delta)\varkappa_2)}{e^{s(\delta\varkappa_1+(1-\delta)\varkappa_2)}}\\
&+\frac{(\phi(1-\delta)\varkappa_2)\varphi(\delta\varkappa_1)}{e^{s((1-\delta)\varkappa_2+\delta\varkappa_1)}}+\frac{(\phi(1-\delta)\varkappa_2)(\varphi(1-\delta)\varkappa_2)}{e^{2s((1-\delta)\varkappa_2)}}.
\end{aligned} \tag{17}$$

Similarly, we have

$$\begin{aligned}
&\phi((1-\delta)\varkappa_1+\delta\varkappa_2)\varphi((1-\delta)\varkappa_1+\delta\varkappa_2)\\
\leq\ & \frac{\phi((1-\delta)\varkappa_1)\varphi((1-\delta)\varkappa_1)}{e^{2s((1-\delta)\varkappa_1)}}+\frac{\phi((1-\delta)\varkappa_1)\varphi(\delta\varkappa_2)}{e^{s(\delta\varkappa_2+(1-\delta)\varkappa_1)}}\\
&+\frac{\phi(\delta\varkappa_2)\varphi((1-\delta)\varkappa_1)}{e^{s((1-\delta)\varkappa_1+\delta\varkappa_2)}}+\frac{\phi(\delta\varkappa_2)\varphi(\delta\varkappa_2)}{e^{2s\delta\varkappa_2}}.
\end{aligned} \tag{18}$$

Adding (17) and (18), we obtain

$$\begin{aligned}
&\phi(\delta\varkappa_1+(1-\delta)\varkappa_2)\varphi(\delta\varkappa_1+(1-\delta)\varkappa_2)\\
&+\phi((1-\delta)\varkappa_1+\delta\varkappa_2)\varphi((1-\delta)\varkappa_1+\delta\varkappa_2)\\
\leq\ & \frac{\phi(\delta\varkappa_1)\varphi(\delta\varkappa_1)}{e^{2s\delta\varkappa_1}}+\frac{\phi(\delta\varkappa_1)\varphi((1-\delta)\varkappa_2)}{e^{s(\delta\varkappa_1+(1-\delta)\varkappa_2)}}\\
&+\frac{(\phi(1-\delta)\varkappa_2)\varphi(\delta\varkappa_1)}{e^{s((1-\delta)\varkappa_2+\delta\varkappa_1)}}+\frac{(\phi(1-\delta)\varkappa_2)(\varphi(1-\delta)\varkappa_2)}{e^{2(s(1-\delta)\varkappa_2)}}\\
&+\frac{\phi((1-\delta)\varkappa_1)\varphi((1-\delta)\varkappa_1)}{e^{2s((1-\delta)\varkappa_1)}}+\frac{\phi((1-\delta)\varkappa_1)\varphi(\delta\varkappa_2)}{e^{s(\delta\varkappa_2+(1-\delta)\varkappa_1)}}\\
&+\frac{\phi(\delta\varkappa_2)\varphi((1-\delta)\varkappa_1)}{e^{s((1-\delta)\varkappa_1+\delta\varkappa_2)}}+\frac{\phi(\delta\varkappa_2)\varphi(\delta\varkappa_2)}{e^{2s\delta\varkappa_2}}.
\end{aligned} \tag{19}$$

Multiplying both sides of (19) by $\delta^{\alpha-1}$and integrating over $[0,1]$, we obtain

$$
\begin{aligned}
&\int_0^1 \delta^{\alpha-1}\phi(\delta\varkappa_1+(1-\delta)\varkappa_2)\varphi(\delta\varkappa_1+(1-\delta)\varkappa_2)d\delta\\
&+\int_0^1 \delta^{\alpha-1}\phi((1-\delta)\varkappa_1+\delta\varkappa_2)\varphi((1-\delta)\varkappa_1+\delta\varkappa_2)d\delta\\
\leq\ &\int_0^1 \delta^{\alpha-1}\left[\frac{\phi(\delta\varkappa_1)\varphi(\delta\varkappa_1)}{e^{2s\delta\varkappa_1}}+\frac{\phi((1-\delta)\,\varkappa_1)\varphi((1-\delta)\,\varkappa_1)}{e^{2s((1-\delta)\varkappa_1)}}\right]d\delta\\
&+\int_0^1 \delta^{\alpha-1}\left[\frac{(\phi(1-\delta)\varkappa_2)(\varphi(1-\delta)\varkappa_2)}{e^{2s((1-\delta)\varkappa_2)}}+\frac{\phi(\delta\varkappa_2)\varphi(\delta\varkappa_2)}{e^{2s\delta\varkappa_2}}\right]d\delta\\
&+\int_0^1 \delta^{\alpha-1}\left[\frac{\phi(\delta\varkappa_1)\varphi((1-\delta)\varkappa_2)}{e^{s(\delta\varkappa_1+(1-\delta)\varkappa_2)}}+\frac{\phi((1-\delta)\,\varkappa_1)\varphi(\delta\varkappa_2)}{e^{s(\delta\varkappa_2+(1-\delta)\varkappa_1)}}\right]\\
&+\int_0^1 \delta^{\alpha-1}\left[\frac{(\phi(1-\delta)\varkappa_2)\varphi(\delta\varkappa_1)}{e^{s((1-\delta)\varkappa_2+\delta\varkappa_1)}}+\frac{\phi(\delta\varkappa_2)\varphi((1-\delta)\,\varkappa_1)}{e^{s((1-\delta)\varkappa_1+\delta\varkappa_2)}}\right]d\delta.
\end{aligned}
$$

By using change of variables of integration, we obtain required result. □

Remark 2.9. *In Theorem 2.3, if we consider* $\alpha = 1$. *Then, we gain [11, Theorem 2.14].*

Remark 2.10. *If we use* $s = 0$ *in Theorem 2.3, then we have [1, Theorem 8 (2.8)].*

Remark 2.11. *If we take* $\phi(\delta\mu) \leq \delta\phi(\mu)$ *and* $s = 0$, *then we have [1, Corollary 2].*

3 Conclusion

In this work, by applying Riemann-Liouville fractional integrals, the authors derived some new Hermite-Hadamard type inequalities for the exponentially subadditive functions. Moreover, the authors showed that the key findings of the paper are extensions of several established results in the field of Hermite-Hadamard type inequalities. It is an interesting and new problem that the forthcoming researchers can obtain different kinds of integral inequalities for the given class of subadditive functions in their future work.

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