Extended Filters of MS-Algebras

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Abstract

For a filter $T$ of an MS-algebra $\mathcal{L}$ and a subset $Z$ of $\mathcal{L}$, a new extension filter of $T$ is introduced, denoted by $E_T(Z)$. Many properties of $E_T(Z)$ are investigated and the lattice structure of the set of all $E_T(Z)$ is studied. A new definition related to $E_T(Z)$ is presented, called fixed filters relative to a subset of $\mathcal{L}$. A generalisation of $E_T(Z)$ is illustrated by introducing the concept of strong filters, notated by $E_T(Z)$. The strong extension $E_T(Z)$ is characterized by the intersection of all strong filters fixed relative to an ideal $\mathcal{L} – \mathfrak{P}$ for a prime filter $\mathfrak{P}$ of $\mathcal{L}$.

Keywords: bounded distributive lattice; MS-algebra; filter; ideal.
1 Introduction

The class $\text{MS}$ of all $\text{MS}$-algebras was first considered by Blyth and Varlet [7]. Their basic goal was to define a common frame to study several similarities between de Morgan algebras and Stone algebras. The class $\text{MS}$ is a subclass of Berman class $\text{K}_{1,1}$ which introduced by Berman in [3]. The theory of filters such as $d_{L}$- filters [1] and $\beta$-filters [12] of $\text{MS}$-algebras has been studied by many authors. In the twenty-first century, many related structures to lattice theory and $\text{MS}$-algebras [10, 11] had a great concern.

In this article, the concept of $E_{T}(Z)$ of a filter $T$ is introduced for a nonempty subset $Z$ in an $\text{MS}$-algebra. We prove that $E_{T}(Z)$ is an extension filter of the filter $T$. A complete distributive lattice is formed by $E_{T}(Z)$. The concept of a fixed filter relative to a subset of an $\text{MS}$-algebra is illustrated. Equivalent conditions were set for the fixed filter relative to a subset of an $\text{MS}$-algebra. The definition of strong fixed filter relative to a subset is introduced. Furthermore, we characterize fixed filters in the terms of strong fixed filters relative to the set $L^{-P}$ for a prime filter $P$.

2 Preliminaries

In the following, we give the basic background to make the paper consistent.

A bounded distributive lattice $(\mathcal{L}; \lor, \land, ^{\circ}, 0, 1)$ together with a unary operation $\lambda \mapsto \lambda^{\circ}$ satisfying,

$$1^{\circ} = 0, \lambda \leq \lambda^{\circ \circ} \quad \text{and} \quad (\lambda \land \mu)^{\circ} = \lambda^{\circ} \lor \mu^{\circ},$$

is called an $\text{MS}$-algebra. Each element in an $\text{MS}$-algebra satisfies the given equalities.

**Proposition 2.1.** [6] Let $\mathcal{L} \in \text{MS}$ and $\lambda, \mu \in \mathcal{L}$. Then;

1. $(\lambda \lor \mu)^{\circ} = \lambda^{\circ} \land \mu^{\circ}$.
2. $(\lambda \lor \mu)^{\circ \circ} = \lambda^{\circ \circ} \lor \mu^{\circ \circ}$.
3. $(\lambda \land \mu)^{\circ \circ} = \lambda^{\circ \circ} \land \mu^{\circ \circ}$.
4. $\lambda^{\circ \circ \circ} = \lambda^{\circ}$.
5. $0^{\circ} = 1$.

For $T \subseteq \mathcal{L}$, $T$ is characterised as a filter provided that $T$ is a sublattice of $\mathcal{L}$ and if $\alpha \in T$, $\omega \in \mathcal{L}$, then $\alpha \lor \omega \in T$. A prime filter $\mathfrak{P}$ is a proper filter satisfying that if $\omega, \tau \in \mathcal{L}$ such that $\omega \lor \tau \in \mathfrak{P}$ then $\omega \in \mathfrak{P}$ or $\tau \in \mathfrak{P}$. Let $\omega \in \mathcal{L}$. We set the notation $[\omega]$ for the principal filter of $\mathcal{L}$ generated by $\omega$ and it is equivalent to the following $[\omega] = \{ \alpha \in \mathcal{L} : \alpha \geq \omega \}$. For a non empty subset $Z \subseteq \mathcal{L}$, the filter $[Z]$ of $\mathcal{L}$ generated by the set $Z$ is defined by

$$[Z] = \{ \lambda \in \mathcal{L} : \lambda \geq z_{1} \land z_{2} \land ... \land z_{n} \quad \text{for} \quad z_{1}, z_{2}, ..., z_{n} \in Z \}. $$

Associating the lattice $\mathcal{L}$ with the distributive property, the symbol $\mathfrak{F}(\mathcal{L})$ stands for the lattice of all filters ordered by inclusion. Obviously, the filter $[1] = \{1\}$ is the smallest member of $\mathfrak{F}(\mathcal{L})$. Also, $[0] = \mathfrak{F}$ is the largest member of $\mathfrak{F}(\mathcal{L})$. We notate the class of all ideals by $\mathfrak{l}(\mathcal{L})$. 

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Theorem 2.1. If \( \mathcal{L} \in \text{MS} \) and \( T, R \in \mathfrak{F}(\mathcal{L}) \). Then
\[
T \triangledown R = \left\{ \lambda \vee \mu : \lambda \in T \text{ and } \mu \in R \right\},
\]
is a member of \( \mathfrak{F}(\mathcal{L}) \).

Proof. Obviously, \( 1 \in T \triangledown R \). For \( \lambda \in T \) and \( \mu \in R \), suppose \( \delta \in \mathcal{L} \) such that \( \delta \geq \lambda \vee \mu \). Therefore \( \delta \in T \) and \( \delta \in R \). Thus, \( \delta = \delta \vee \delta \in T \triangledown R \).

Let \( \delta, \gamma \in T \triangledown R \). Then there exist \( \lambda_1, \lambda_2 \in T \) and \( \mu_1, \mu_2 \in R \) such that \( \delta = \lambda_1 \vee \mu_1 \) and \( \gamma = \lambda_2 \vee \mu_2 \). We have that
\[
\delta \wedge \gamma = (\lambda_1 \vee \mu_1) \wedge (\lambda_2 \vee \mu_2)
\]
\[
\quad = \left[ (\lambda_1 \vee \mu_1) \wedge \lambda_2 \right] \vee \left[ (\lambda_1 \vee \mu_1) \wedge \mu_2 \right]
\]
\[
\quad = \left[ (\lambda_1 \wedge \lambda_2) \vee (\mu_1 \wedge \lambda_2) \right] \vee \left[ (\lambda_1 \wedge \mu_2) \vee (\mu_1 \wedge \mu_2) \right] \in T \triangledown R,
\]
since \( \lambda_1 \wedge \lambda_2 \in T \) and \( \mu_1 \wedge \mu_2 \in R \). Hence \( T \triangledown R \in \mathfrak{F}(\mathcal{L}) \). \( \square \)

Theorem 2.2. For \( \mathcal{L} \in \text{MS} \). Let \( J, K \in \mathbb{I}(\mathcal{L}) \). Define
\[
J \triangledown I K = \left\{ z : z \leq \lambda \wedge \mu : \lambda \in J \text{ and } \mu \in K \right\}.
\]
Then \( J \triangledown I K \in \mathbb{I}(\mathcal{L}) \).

Proof. Clearly, \( 0 \in J \triangledown I K \). For \( \lambda \in J \) and \( \mu \in K \), suppose \( \delta \in \mathcal{L} \), such that \( \delta \leq \lambda \wedge \mu \). Obviously, \( \delta \in J \triangledown I K \). Let \( \delta, \gamma \in J \triangledown I K \). It follows that \( \delta \leq \lambda_1 \wedge \mu_1 \) and \( \delta \leq \lambda_2 \wedge \mu_2 \) for \( \lambda_1, \lambda_2 \in J \) and \( \mu_1, \mu_2 \in K \). Then,
\[
\delta \triangledown \gamma \leq (\lambda_1 \wedge \mu_1) \triangledown (\lambda_2 \wedge \mu_2)
\]
\[
\quad \leq \left[ (\lambda_1 \wedge \mu_1) \triangledown \lambda_2 \right] \wedge \left[ (\lambda_1 \wedge \mu_1) \triangledown \mu_2 \right]
\]
\[
\quad \leq \left[ (\lambda_1 \triangledown \lambda_2) \wedge (\mu_1 \triangledown \lambda_2) \right] \wedge \left[ (\lambda_1 \triangledown \mu_2) \wedge (\mu_1 \triangledown \mu_2) \right] \in J \triangledown I K,
\]
since \( \lambda_1 \triangledown \lambda_2 \in J \) and \( \mu_1 \triangledown \mu_2 \in K \). \( \square \)

Theorem 2.3. [9] For \( \mathcal{L} \in \text{MS} \). The set \( \mathcal{L} - \mathfrak{P} \in \mathbb{I}(\mathcal{L}) \) providing that \( \mathfrak{P} \) is a prime filter of \( \mathcal{L} \).

For details of MS-algebras, [2] highlighted many aspects of the variety MS. In [5], the subvarieties of MS were determined. Also, many constructions and substructures of MS-algebras were presented in [4, 8]. Throughout the paper we use the symbol \( \mathfrak{L} \) for an MS-algebra.

3 Extended Filter

For \( T \in \mathfrak{F}(\mathcal{L}) \) and a nonempty subset \( Z \) of \( \mathcal{L} \), define
\[
E_T(Z) = \left\{ \lambda \in \mathcal{L} : \lambda \vee z^\circ \in T \text{ for every } z \in Z \right\}.
\]
Theorem 3.1. For $T \in \mathcal{F}(\mathcal{L})$, the set $E_T(Z)$ is a filter of $\mathcal{L}$ containing $T$.

Proof. Obviously, $1 \in E_T(Z)$. Assume that $\lambda \in E_T(Z)$ and $\mu \in \mathcal{L}$ satisfying $\lambda \leq \mu$. We have that $\mu \lor z^{00} \geq \lambda \lor z^{00}$. Therefore $\mu \lor z^{00} \in T$. Then $\mu \in E_T(Z)$. Assume that $\lambda, \mu \in E_T(Z)$. Since $(\lambda \land \mu) \lor z^{00} = (\lambda \lor z^{00}) \land (\mu \lor z^{00}) \in T$, then $\lambda \land \mu \in T$. Clearly, $T \subseteq E_T(Z)$.

We call $E_T(Z)$ an extended filter of $T$. The following theorem encapsulates many characterisations of $E_T(Z)$.

Lemma 3.1. Let $T \in \mathcal{F}(\mathcal{L})$. For any nonempty subset $Z$ of $\mathcal{L}$, we have:

1. If $Z$ is contained in the subset $W$, then $E_T(W) \subseteq E_T(Z)$.
2. If $R$ is a filter contains $T$, then $E_T(Z) \subseteq E_R(Z)$.
3. If $T$ contains each element of $Z$, then $E_T(Z) = \mathcal{L}$.
4. If $0 \in Z$, then $E_T(Z) = \mathcal{L}$ implies that $Z \subseteq T$.
5. If $T \subseteq Z$ and $z^{00} = 0$ for some $z \in Z$, then $E_T(Z) \cap Z = T$.
6. If $\alpha^{00} = 0$ for some $\alpha \in E_T(Z)$, then $E_T(E_T(Z)) \cap E_T(Z) = T$.
7. $E_T(Z) = E_T([Z])$.
8. $E_{E_T(Z)}(W) = E_{E_T(W)}(Z)$.

Proof.

1. Assume that $Z \subseteq W$. If $\lambda \in E_T(W)$, then $\lambda \lor w^{00} \in T$ for every $w \in W$. It follows that $\lambda \lor z^{00} \in T$ for every $z \in Z \subseteq W$. Hence $\lambda \in E_T(Z)$.
2. Suppose $T \subseteq R$ and $\lambda \in E_T(Z)$. This implies that $\lambda \lor z^{00} \in T \subseteq R$ for every $z \in Z$. Thus $\lambda \in E_R(Z)$. Hence $E_T(Z) \subseteq E_R(Z)$.
3. Let $Z \subseteq T$. Obviously, $E_T(Z) \subseteq \mathcal{L}$. Conversely, suppose $\lambda \in \mathcal{L}$ and $z \in Z$. Since $z \in T$ and $\lambda \lor z^{00} \geq z^{00} \geq z$, then $\lambda \lor z^{00} \in T$. We conclude that $E_T(Z) = \mathcal{L}$.
4. Assume that $\lambda \in Z$ and $E_T(Z) = \mathcal{L}$. Then $\lambda \lor z^{00} \in T$ for every $z \in Z$. Hence $\lambda = \lambda \lor 0^{00} \in T$.
5. We have that $T \subseteq E_T(Z) \cap Z$. Conversely, let $\lambda \in E_T(Z) \cap Z$. We get that $\lambda \in Z$ and $\lambda \in E_T(Z)$. Then $\lambda \lor z^{00} \in T$ for every $z \in Z$. Thus $\lambda = \lambda \lor 0 \in T$. This implies that $E_T(Z) \cap Z \subseteq T$. Hence $E_T(Z) \cap Z = T$.
6. Follows directly from (5).
7. By (1), $E_T([Z]) \subseteq E_T(Z)$. Conversely, suppose that $\lambda \in E_T(Z)$, then $\lambda \lor z^{00} \in T$ for every $z \in Z$. Let $p \in [Z]$. It follows that $p \geq z_1 \land z_2 \land \ldots \land z_n$ for some $z_1, \ldots, z_n \in Z$. Then $\lambda \lor p^{00} \geq \lambda \lor (z_1^{00} \land \ldots \land z_n^{00}) \geq (\lambda \lor z_1^{00}) \land \ldots \land (\lambda \lor z_n^{00}) \in T$.

Hence $E_T(Z) = E_T([Z])$. 

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Remark 3.1.

(1) The converse of (3) is not necessarily true. For example, set \( \mathcal{L} = \{0 \leq \lambda \leq \mu \leq \gamma \leq 1\} \), such that \( \mu = \mu^0 = \lambda^0, \gamma^0 = 0 = 1^0, 0^0 = 1 \). Clearly, \((L, \cup) \in \text{MS} \). Suppose that \( T = \{\mu\} = \{\mu, \gamma, 1\}\) and \( Z = \{\lambda, \gamma\} \not\subseteq T \). Then \( E_T(Z) = \mathcal{L} \). So the condition \( 0 \in Z \) in (4) is necessary.

(2) The set \( Z \) is not necessarily a subset of \( E_T(E_T(Z)) \). For example, we obtain in the previous example that \( E_T(Z) = \mathcal{L} \) and \( E_T(E_T(Z)) = E_T(\mathcal{L}) = T \).

For \( T \in \mathfrak{F}(\mathcal{L}) \) and \( Z \subseteq \mathcal{L} \), we use the following notations:

\[
\mathbb{E}(Z) = \left\{ E_T(Z); \ T \in \mathfrak{F}(\mathcal{L}) \right\},
\]

\[
\mathbb{E}_T = \left\{ E_T(Z); \ Z \subseteq \mathcal{L} \right\}.
\]

Proposition 3.1. If \( T \in \mathfrak{F}(\mathcal{L}) \), then \( T \) is a member of \( \mathbb{E}_T \). Moreover, \( T \) is the smallest element in \( \mathbb{E}_T \).

Proof. It is easy to prove that \( T = E_T(\{0\}) \), thus \( T \in \mathbb{E}_T \). Also, for every non empty \( Z \subseteq \mathcal{L} \) we have that \( T \subseteq E_T(Z) \). Hence \( T \) is the smallest element in \( \mathbb{E}_T \). □

In the next lemma, basic properties of \( \mathbb{E}(Z) \) and \( \mathbb{E}_T \) are investigated seeking for constructing a new lattice.

Lemma 3.2. Let \( \mathcal{L} \in \text{MS} \). For nonempty subsets \( Z \) and \( W \) of \( \mathcal{L} \), we have;

1. \( \bigcup_{i \in I} E_T(Z_i) \subseteq E_T(\bigcap_{i \in I} Z_i) \).
2. \( E_T(\bigcup_{i \in I} Z_i) \subseteq \bigcap_{i \in I} E_T(Z_i) \).
3. \( E_T(Z) \cap E_T(W) = E_T(Z \cup W) \).

Proof.

1. Obviously, \( \bigcap_{i \in I} Z_i \subseteq Z_i \) for every \( i \in I \). Then \( E_T(Z_i) \subseteq E_T(\bigcap_{i \in I} Z_i) \) for every \( i \in I \). Hence, \( \bigcup_{i \in I} E_T(Z_i) \subseteq E_T(\bigcap_{i \in I} Z_i) \).

2. Clearly, \( Z_i \subseteq \bigcup_{i \in I} Z_i \) for every \( i \in I \). By Lemma 3.1 (1), \( E_T(\bigcup_{i \in I} Z_i) \subseteq E_T(Z_i) \) for every \( i \in I \). Hence, \( E_T(\bigcup_{i \in I} Z_i) \subseteq \bigcap_{i \in I} E_T(Z_i) \).
(3) Let \( \lambda \in E_T(Z \cup W) \). Then \( \lambda \lor \mu^\circ \in T \) for every \( \mu \in Z \cup W \). Thus, \( \lambda \lor z^\circ \in T \) and \( \lambda \lor w^\circ \in T \) and for every \( z \in Z \) and every \( w \in W \). So, \( \lambda \in E_T(Z) \) and \( \lambda \in E_T(W) \). This implies that \( \lambda \in E_T(Z) \lor E_T(W) \).

Conversely, assume that \( e \in E_T(Z) \lor E_T(W) \). Then, \( e = \lambda \lor \mu \) for some \( \lambda \in E_T(Z) \) and \( \mu \in E_T(W) \). Let \( z \in Z \) and \( w \in W \). Then,

\[
e \lor z^\circ = (\lambda \lor \mu) \lor z^\circ = \mu \lor (\lambda \lor z^\circ) \in T \quad \text{since} \quad \lambda \lor z^\circ \in T.
\]

Similarly, \( e \lor w^\circ \in T \). Hence \( \lambda \in E_T(Z \cup W) \).

\[\square\]

**Theorem 3.2.** Let \( T \in \mathfrak{F}(\mathfrak{L}) \). Let \( Z \) and \( W \) be nonempty subsets of \( \mathfrak{L} \). Then,

1. \( E_T(\emptyset) = T = E_T(\{0\}) \).
2. \( E_T(\{1\}) = \mathfrak{L} = E_T(T) \).
3. \( E_T(Z) \cap E_T(W) = E_T(Z \cup W) \).
4. If \( E_T(Z \cap W) \subseteq E_T(Z) \lor E_T(W) \), then \( E_T(Z \cap W) = E_T(Z) \lor E_T(W) \).

**Proof.**

1. We have that \( T \subseteq E_T(\mathfrak{L}) \). On the other hand, let \( \lambda \in E_T(\emptyset) \). Thus \( \lambda \lor a^\circ \in T \) for every \( a \in \emptyset \). Since \( \emptyset \) is bounded, we get that \( \lambda = \lambda \lor 0^\circ \in T \). We can easily see that \( E_T(\{0\}) = T \).

2. Clearly, \( \mathfrak{L} = E_T(T) \). We only need to prove that \( \mathfrak{L} = E_T(\{1\}) \). Assume that \( \lambda \in \mathfrak{L} \), then \( \lambda \lor 1^\circ = 1 \in T \).

3. \( Z, W \subseteq Z \cup W \) imply that \( E_T(Z \cup W) \subseteq E_T(Z) \) and \( E_T(Z \cup W) \subseteq E_T(W) \). So, \( E_T(Z \cup W) \subseteq E_T(Z) \cap E_T(W) \). Conversely, let \( \lambda \in E_T(Z) \cap E_T(W) \). Then \( \lambda \in E_T(Z) \) and \( \lambda \in E_T(W) \). It follows that \( \lambda \lor z^\circ \in T \) for every \( z \in Z \) and \( \lambda \lor w^\circ \in T \) for every \( w \in W \).

Therefore \( \lambda \lor a^\circ \in T \) for every \( a \in Z \cup W \). Hence \( \lambda \in E_T(Z \cup W) \).

4. Since \( Z \cap W \subseteq Z, W \), then \( E_T(Z), E_T(W) \subseteq E_T(Z \cap W) \). Hence, \( E_T(Z) \lor E_T(W) \subseteq E_T(Z \cap W) \).

\[\square\]

**Corollary 3.1.** For \( T \in \mathfrak{F}(\mathfrak{L}) \). Assume that \( E_T(Z \cap W) \subseteq E_T(Z) \lor E_T(W) \) for any two subsets \( Z \) and \( W \) of \( \mathfrak{L} \). Then \( (E_T; \lor, E_T(\{0\}), E_T(\{1\})) \) is a bounded distributive lattice.

**Remark 3.2.** Obviously, if \( \mathfrak{L} \) is a complete lattice, then \( (E_T; \lor, E_T(\{0\}), E_T(\{1\})) \) is also a complete lattice.

**Theorem 3.3.** If \( Z \) is a subset of \( \mathfrak{L} \), then \( (\mathfrak{E}(Z); \lor, E_{\{1\}}(Z), E_{\{0\}}(Z)) \) is a bounded distributive lattice. Moreover, \( \mathfrak{E}(Z) \) is a complete lattice providing that \( \mathfrak{L} \) is a complete lattice.
Proof. For a subset $Z$ of $\mathcal{L}$ and $T \in \mathfrak{F}(\mathcal{L})$, we show that $E \bigcap_{i \in I} T_i(Z) = \bigcap_{i \in I} E_{T_i}(Z)$. We have that $\bigcap_{i \in I} T_i \subseteq T_i$ for every $i \in I$. By Lemma 3.1 (2), $E \bigcap_{i \in I} T_i(Z) \subseteq E_{T_i}(Z)$ for every $i \in I$. Then $E \bigcap_{i \in I} T_i(Z) \subseteq \bigcap_{i \in I} E_{T_i}(Z)$.

Conversely, let $\lambda \in \bigcap_{i \in I} E_{T_i}(Z)$. Then $\lambda \in E_{T_i}(Z)$ for every $i \in I$. This implies that $\lambda \lor z^{\circ\circ} \in T_i$ for every $z \in Z$ for every $i \in I$. Then $\lambda \lor z^{\circ\circ} \in \bigcap_{i \in I} T_i$ for every $z \in Z$. Therefore $\lambda \in E \bigcap_{i \in I} T_i(Z)$. Hence $E \bigcap_{i \in I} T_i(Z) = \bigcap_{i \in I} E_{T_i}(Z)$.

We also need to show that $E_{T\cap R}(Z) = E_T(Z) \cap E_R(Z)$. By Theorem 2.1, we have that $E_T(Z) \cap E_R(Z) = \{\lambda \lor \mu; \lambda \in E_T(Z), \mu \in E_R(Z)\}$ is a filter of $\mathcal{L}$. For every $i \in T$ and $r \in R$ we have that $t, r \leq t \lor r$. Then $t \lor r \in T, R$. Therefore $T \cap R \subseteq T, R$ and then $E_{T \cap R}(Z) \subseteq E_T(Z), E_R(Z)$. Thus $E_{T \cap R}(Z) \subseteq E_T(Z) \cap E_R(Z)$. Conversely, assume that $e \in E_T(Z) \cap E_R(Z)$, therefore $e = \lambda \lor \mu$ for some $\lambda \in E_T(Z)$ and $\mu \in E_R(Z)$, then for every $z \in Z$ we have,

$$e \lor z^{\circ\circ} = (\lambda \lor \mu) \lor z^{\circ\circ} = (\lambda \lor z^{\circ\circ}) \lor (\mu \lor z^{\circ\circ}) \in T \cap R.$$ 

Hence $e \in E_{T \cap R}(Z)$. If $\mathcal{L}$ is a complete, then $(E_T, \lor, E(R), \{0\}, E(\{1\}))$ is complete. \[\square\]

**Definition 3.1.** A filter $T$ of $\mathcal{L}$ is said to be fixed relative to a subset $Z$ of $\mathcal{L}$ if $E_T(Z) = T$.

We denote the class of all fixed filters relative to subset $Z$ of $\mathcal{L}$ by $\Delta_Z$. The following example illustrates the concept of a fixed filter relative to a subset of $\mathcal{L}$.

**Example 3.1.** Let $\mathcal{L} = \{0 \leq \mu \leq \delta \leq 1\}$ such that $\mu = \mu^{\circ\circ}$, $\delta^{\circ} = 0$, $0^{\circ} = 1$. Obviously, $(\mathcal{L}, \circ\circ) \in MS$. Suppose that $T = [\mu] = \{\mu, \delta, 1\}$ and $Z = \{\mu, 0\}$. Obviously, $E_T(Z) = T$. Then $T$ is fixed relative to $Z$. Suppose that $C = \{\delta\}$. Thus $E_T(C) = \mathcal{L}$. Hence $T$ is not fixed relative to $C$.

**Proposition 3.2.** Let $T \in \mathfrak{F}(\mathcal{L})$ and $Z \in \mathcal{L}$. The following statements are equivalent:

1. If $\lambda^{\circ\circ} = 0$ for some $\lambda \in E_T(Z)$, then $E_T(E_T(Z)) = \mathcal{L}$.
2. $T$ is fixed relative to a subset $Z$.
3. $T$ is fixed relative to a subset $\{Z\}$.

**Proof.** By Lemma 3.1 (7), $E_T(Z) = T$ is equivalent to $E_T(\{Z\}) = T$. Then (2) if and only if (3). Assume the condition of (2). We get that $E_T(E_T(Z)) = E_T(T) = \mathcal{L}$. Thus (2) implies (1). Consider (1). Therefore $E_T(E_T(Z)) = \mathcal{L}$. By Lemma 3.1 (6), $\mathcal{L} \cap E_T(Z) = T$. Thus $E_T(Z) = T$. Hence, (1) implies (2). \[\square\]

**Proposition 3.3.** For a maximal filter $M$ of $\mathcal{L}$, $M$ is fixed relative to $Z$ provided that $E_M(Z)$ is a proper filter of $\mathcal{L}$.

**Proof.** Since $M \subseteq E_M(Z)$ and $E_M(Z) \neq \mathcal{L}$, then $M = E_M(Z)$. \[\square\]

**Proposition 3.4.** Let $T \in \mathfrak{F}(\mathcal{L})$ and let $Z, W \subseteq \mathcal{L}$. If $Z \subseteq W$ and $T$ is fixed relative to $Z$, then $T$ is fixed relative to $W$. 465
Proof. Let \( Z \subseteq W \). Then \( E_T(W) \subseteq E_T(Z) = T \). Therefore \( E_T(W) = T \). Hence \( T \) is fixed relative to \( W \).

**Proposition 3.5.** For \( Z \subseteq L \), the set \( \Delta_Z \) is a meet semi lattice of \((E(Z), \cap)\).

Proof. Clearly, \( \Delta_Z \) is an ordered subset of \( E(Z) \) by restricting the relation \( \leq \) to \( \Delta_Z \). By Theorem 3.3, \( E_T \cap R(Z) = E_T(Z) \cap E_R(Z) = T \cap R \in \Delta_Z \) for \( T, R \in \Delta_Z \).

4 Strong Extensions

In this section, we go further by defining the concept of strong fixed filter \( E_T(\kappa) \) relative to an ideal \( \kappa \) of an MS-algebra. We notate the class of all prime filters by \( \text{Spec}(L) \). For \( T \in \mathfrak{F}(L) \), define

\[
E_T(\kappa) = \left\{ \alpha \in L : \alpha \lor a^{\infty} \in T, \text{ for some } a \in \kappa \right\},
\]

for an ideal \( \kappa \). Obviously, \( E_T(\kappa) \subseteq E_T(\kappa) \). So, we have \( T \subseteq E_T(\kappa) \subseteq E_T(\kappa) \). Thus \( E_T(\kappa) \) is an extension of both \( T \) and \( E_T(\kappa) \).

**Theorem 4.1.** If \( L \in \text{MS}, \kappa \in \mathfrak{l}(L) \) and \( T \in \mathfrak{F}(L) \). Then \( E_T(\kappa) \) is a filter of \( L \).

Proof. We see that \( 1 \in E_T(\kappa) \), as \( 1 = 1 \lor 0^{\infty} \). Assume that \( \lambda \in E_T(\kappa) \). Then \( \lambda \lor a^{\infty} \in T \) for some \( a \in \kappa \). Let \( \mu \in L \) satisfying that \( \lambda \leq \mu \). Then \( \mu \lor a^{\infty} \geq \lambda \lor a^{\infty} \in T \). Thus \( \mu \in E_T(\kappa) \). If \( \lambda, \mu \in E_T(\kappa) \), then \( \lambda \lor a^{\infty} \in T \) and \( \mu \lor b^{\infty} \in T \) for some \( a, b \in \kappa \). We have

\[
(\lambda \land \mu) \lor (a \lor b)^{\infty} = (\lambda \land \mu) \lor (a^{\infty} \lor b^{\infty}) = [(\lambda \lor a^{\infty}) \lor b^{\infty}] \land [(\mu \lor b^{\infty}) \lor a^{\infty}].
\]

Since \( (\lambda \lor a^{\infty}) \lor b^{\infty} \geq \lambda \lor a^{\infty} \) and \( (\mu \lor b^{\infty}) \lor a^{\infty} \geq \mu \lor b^{\infty} \). We conclude that \( [(\lambda \lor a^{\infty}) \lor b^{\infty}] \land [(\mu \lor b^{\infty}) \lor a^{\infty}] \in T \). Hence \( \lambda \land \mu \in E_T(\kappa) \).

The inclusion \( E_T(\kappa) \subseteq E_T(\kappa) \) is proper as shown in the next example.

**Example 4.1.** Consider \( L \) with the following Hasse diagram:

![Hasse diagram](image)
Define a unary operation \( ^o \) on \( \mathcal{L} \) by \( \lambda^o = t, \beta^o = z^o = t^o = u, u^o = \beta, 1^o = 0, 0^o = 1 \). Then \((\mathcal{L}, ^o) \) is MS. Take \( T = [\beta] = \{ \beta, z, u, 1 \} \) and \( \kappa = (t) = \{ 0, t \} \). Then,

\[
E_T(\{t\}) = \left\{ n \in \mathcal{L} : n \lor 0^o \in T \text{ and } n \lor t^o \in T \right\} = \left\{ n \in \mathcal{L} : n \in T \text{ and } n \lor \beta \in T \right\} = \{ \beta, z, u, 1 \} = [\beta].
\]

\[
\overline{E_T}(\{t\}) = \left\{ n \in \mathcal{L} : n \lor 0^o \in T \text{ or } n \lor t^o \in T \right\} = \left\{ n \in \mathcal{L} : n \in T \text{ or } n \lor \beta \in T \right\} = \{ 0, t, \lambda, \beta, z, u, 1 \}.
\]

**Lemma 4.1.** Let \( \mathcal{L} \in \text{MS} \), \( T, R \in \mathfrak{F}(\mathcal{L}) \) and \( \kappa, \kappa_1, \kappa_2 \in \mathfrak{I}(\mathcal{L}) \). Then;

1. \( \kappa_1 \subseteq \kappa_2 \) implies that \( \overline{E_T}(\kappa_1) \subseteq \overline{E_T}(\kappa_2) \).
2. \( T \subseteq R \) implies that \( E_T(\kappa) \subseteq E_R(\kappa) \).
3. \( E_T(\kappa) \cap \overline{E_R(\kappa)} = \overline{E_{T \cap R}}(\kappa) \).
4. \( E_T(\kappa_1) \cap \overline{E_T}(\kappa_2) = \overline{E_T}(\kappa_1 \cap \kappa_2) \).
5. \( \overline{E_T}(\kappa) = \overline{E_{E_T(\kappa)}}(\kappa) \).

**Proof.**

1. If \( \alpha \in \overline{E_T}(\kappa_1) \), then \( \alpha \lor a^o \in T \) for some \( a \in \kappa_1 \subseteq \kappa_2 \). Thus \( \alpha \in \overline{E_T}(\kappa_2) \).
2. Suppose that \( \lambda \in \overline{E_T}(\kappa) \). We get that \( \lambda \lor a^o \in T \subseteq R \) for some \( a \in \kappa \). Consequently, \( \lambda \in E_R(\kappa) \).
3. We have that \( E_{T \cap R}(\kappa) \subseteq E_T(\kappa) \) and \( \overline{E_{T \cap R}(\kappa)} \subseteq \overline{E_R(\kappa)} \).
   Since \( T \cap R \subseteq T, R \), then \( E_{T \cap R}(\kappa) \subseteq E_T(\kappa) \cap E_R(\kappa) \). If \( \lambda \in \overline{E_T(\kappa) \cap \overline{E_R(\kappa)}} \), then \( \lambda \in \overline{E_T(\kappa)} \) and \( \lambda \in E_R(\kappa) \). Therefore \( \lambda \lor a^o \in T \) and \( \lambda \lor b^o \in R \) for some \( a, b \in \kappa \). These imply that \( \lambda \lor (a \lor b)^o = \lambda \lor a^o \lor b^o \geq \lambda \lor b^o \lor a^o \). Then \( \lambda \lor (a \lor b)^o \in T \cap R \). Thus \( \lambda \in \overline{E_{T \cap R}(\kappa)} \). We conclude that \( \overline{E_T}(\kappa) \cap \overline{E_R(\kappa)} = \overline{E_{T \cap R}}(\kappa) \).
4. As \( \overline{E_T}(\kappa_1 \cap \kappa_2) \subseteq \overline{E_T}(\kappa_1), E_T(\kappa_2) \), then \( E_T(\kappa_1 \cap \kappa_2) \subseteq \overline{E_T}(\kappa_1) \cap \overline{E_T}(\kappa_2) \). Conversely, let \( \lambda \in E_T(\kappa_1) \cap E_T(\kappa_2) \). Then \( E_T(\kappa_1) \) and \( E_T(\kappa_2) \). It follows that \( \lambda \lor a^o \in T \) for some \( a \in \kappa_1 \) and \( \lambda \lor b^o \in T \) for some \( b \in \kappa_2 \). Then,
   \[
   \lambda \lor (a \lor b)^o = (\lambda \lor a^o) \lor (\lambda \lor b^o) \in T.
   \]
   Since \( a \land b \in \kappa_1 \cap \kappa_2 \), then \( \lambda \in \overline{E_T}(\kappa_1 \cap \kappa_2) \).
5. Since \( T \subseteq \overline{E_T}(\kappa) \), by (2), we get that \( \overline{E_T}(\kappa) \subseteq \overline{E_{E_T(\kappa)}}(\kappa) \). Conversely, let \( \lambda \in \overline{E_{E_T(\kappa)}}(\kappa) \).
   Then \( \lambda \lor a^o \in \overline{E_T}(\kappa) \) for some \( a \in \kappa \). Therefore, \( (\lambda \lor a^o) \lor b^o \in T \) for some \( a, b \in \kappa \). Then \( \lambda \lor (a \lor b)^o \in T \). As \( a \lor b \in \kappa \), we get that \( \lambda \in E_T(\kappa) \).
Lemma 4.2. Let $T \in \mathfrak{F}(\mathcal{L})$ and let $\Lambda$ be a chain of members of $\mathbb{I}(\mathcal{L})$. Then

$$E_T(\bigcup_{\kappa \in \Lambda} \kappa) = \bigcup_{\kappa \in \Lambda} E_T(\kappa).$$

Proof. Clearly, $\bigcup_{\kappa \in \Lambda} \kappa$ is an ideal of $\mathcal{L}$. For each $\kappa \in \Lambda$, $\kappa \subseteq \bigcup_{\kappa \in \Lambda} \kappa$. This implies that $E_T(\kappa) \subseteq E_T(\bigcup_{\kappa \in \Lambda} \kappa)$. Then, $E_T(\kappa) \subseteq E_T(\bigcup_{\kappa \in \Lambda} \kappa)$. Conversely, let $\lambda \in E_T(\bigcup_{\kappa \in \Lambda} \kappa)$. Thus $\lambda \lor a^{oo} \in T$ for some $a \in \bigcup_{\kappa \in \Lambda} \kappa$. So, there exists $\kappa \in \Lambda$ such that $a \in \kappa$ and $\lambda \lor a^{oo} \in T$. Therefore $\lambda \in E_T(\kappa)$ for some $\kappa \in \Lambda$. It follows that $E_T(\bigcup_{\kappa \in \Lambda} \kappa) \subseteq \bigcup_{\kappa \in \Lambda} E_T(\kappa)$. Hence the claim is true.

Theorem 4.2. If $T \in \mathfrak{F}(\mathcal{L})$ and $\kappa \in \mathbb{I}(\mathcal{L})$, then,

$$E_T(\kappa) = \bigcap \{ E_T(\mathcal{L} - \mathfrak{P}) ; \mathfrak{P} \in \text{Spec}(\mathcal{L}) , \kappa \subseteq \mathcal{L} - \mathfrak{P} \}.$$

Proof. We have $E_T(\kappa) \subseteq E_T(\mathcal{L} - \mathfrak{P})$ for every $\mathfrak{P} \in \text{Spec}(\mathcal{L})$. Since $\kappa \subseteq \mathcal{L} - \mathfrak{P}$, then $E_T(\kappa) \subseteq \bigcap \{ E_T(\mathcal{L} - \mathfrak{P}) ; \mathfrak{P} \in \text{Spec}(\mathcal{L}) ; \kappa \subseteq \mathcal{L} - \mathfrak{P} \}$. On the other hand, by contrapositive we prove that $a \notin \bigcap \{ E_T(\mathcal{L} - \mathfrak{P}) ; \mathfrak{P} \in \text{Spec}(\mathcal{L}) ; \kappa \subseteq \mathcal{L} - \mathfrak{P} \}$. Consider $\Gamma = \{ J \in \mathcal{L} ; \kappa \subseteq J \land a \notin E_T(J) \}$. Obviously, $\kappa \in \Gamma$ so, $\Gamma \neq \emptyset$. Let $\Lambda$ be a chain of members of $\Gamma$ and $G = \bigcup_{J \in \Lambda} J$. By Lemma 4.2, $E_T(G) = \bigcup_{J \in \Gamma} E_T(J)$. Also, $\kappa \subseteq G$. Let $a \notin E_T(\kappa)$.

We show that there exists $\mathfrak{P} \in \text{Spec}(\mathcal{L})$ satisfying that $\kappa \subseteq \mathcal{L} - \mathfrak{P}$ and $a \notin E_T(\mathcal{L} - \mathfrak{P})$. Now, $a \notin E_T(J)$ for all $J \in \Lambda$, implies that $a \notin E_T(G)$. Therefore $E_T(G)$ is an upper bound of $\Lambda$. By Zorn’s Lemma, $\Gamma$ has a maximal element $J_0$. Then $a \notin E_T(J_0)$. So, $\mathcal{L} \neq E_T(J_0)$. Equivalently, $J_0 \neq \mathcal{L}$. Consider $\mathfrak{P}_0 = \mathcal{L} - J_0$. We show that $\mathfrak{P}_0 \in \text{Spec}(\mathcal{L})$. Clearly 1 $\in \mathfrak{P}_0$. Let $\lambda \in \mathfrak{P}_0$ and $\mu \geq \lambda$. Then $\mu \notin J_0$. So, $\mu \in \mathfrak{P}_0$. Suppose that $\lambda, \mu \in \mathfrak{P}_0$. This implies that $\lambda^{oo}, \mu^{oo} \notin J_0$. So, $J_0 \subseteq \langle J_0 \cup \{ \lambda^{oo} \} \rangle$. Since $J_0$ is a maximal element of $\Gamma$, then $J_0 \cup \{ \lambda^{oo} \} \notin \Gamma$. We have $\kappa \subseteq J_0 \subseteq \langle J_0 \cup \{ \lambda^{oo} \} \rangle \notin \Gamma$ and $\kappa \subseteq J_0 \subseteq \langle J_0 \cup \{ \mu^{oo} \} \rangle \notin \Gamma$. Therefore,

$$a \in E_T(J_0 \cup \{ \lambda^{oo} \}) \quad \text{and} \quad a \notin E_T(J_0 \cup \{ \mu^{oo} \}).$$

It follows that there exists $b \in (J_0 \cup \{ \lambda^{oo} \}) \setminus (J_0 \cup \{ \mu^{oo} \})$ such that $a \lor b^{oo} \in T$. That is, there exist $\lambda_1, \mu_1 \in J_0$ such that $b \leq \lambda_1 \lor \lambda^{oo}$ and $b \leq \mu_1 \lor \mu^{oo}$. Let $z = \lambda_1 \lor \lambda^{oo}$ and $b \leq z \lor \mu^{oo}$. Therefore $a \lor b^{oo} \leq a \lor z^{oo} \lor \lambda^{oo}$ and $b \lor b^{oo} \leq a \lor z^{oo} \lor \mu^{oo}$. It follows that $a \lor z^{oo} \lor \lambda^{oo}, a \lor z^{oo} \lor \mu^{oo} \in T$. We get directly that $(a \lor z^{oo} \lor \lambda^{oo}) \land (a \lor z^{oo} \lor \mu^{oo}) \in T$. Then $(a \lor z^{oo}) \lor (\lambda^{oo} \land \mu^{oo}) \in T$. Thus $\lambda \land \mu \in \mathfrak{P}_0$. Otherwise, if $\lambda \land \mu \notin \mathfrak{P}_0$, then $\lambda \land \mu \in J_0$ implies that $a \lor z^{oo} \in E_T(J_0) = E_T(J_0)(J_0)$. Therefore, $a \in E_T(J_0)$, which is a contradiction. Then $\mathfrak{P}_0$ is a filter.

It remains to prove that $\mathfrak{P}_0$ is prime. If $a \lor b \notin \mathfrak{P}_0$. Then $a \lor b \notin J_0$. Thus $a \notin J_0$ or $b \notin J_0$. We conclude that $a \notin \mathfrak{P}_0$ or $b \notin \mathfrak{P}_0$. Thus $\mathfrak{P}_0$ is prime.

Moreover, $\kappa \subseteq \kappa_0 = \mathcal{L} - \mathfrak{P}_0$ and $a \notin E_T(J_0) = E_T(\mathcal{L} - \mathfrak{P}_0)$. Hence, $\mathfrak{P}_0 \in \text{Spec}(\mathcal{L})$. Therefore, $E_T(\mathcal{L} - \mathfrak{P}) \subseteq E_T(\kappa)$ and the proof is complete.

Corollary 4.1. Let $\mathcal{L} \in \text{MS}, \mu \in \mathcal{L}$ and $T \in \mathfrak{F}(\mathcal{L})$. Then,

$$E_T(\{ \mu \}) = \bigcap \{ E_T(\mathcal{L} - \mathfrak{P}) ; \mathfrak{P} \in \text{Spec}(\mathcal{L}), \mu \subseteq \mathcal{L} - \mathfrak{P} \}.$$
Proof. We prove that $E_T([\mu]) = E_T(\{\mu\})$. Clearly, $E_T(\{\mu\}) \subseteq E_T([\mu])$. Let $\lambda \in E_T([\mu])$. Then there exists $b \in (\mu]$ such that $\lambda \lor b^\circ \in T$. Thus $\lambda \lor b^\circ \leq \lambda \lor \mu^\circ \in T$. Then $\lambda \in E_T(\{\mu\})$.

From Theorem 4.2, $E_T(\{\mu\}) = \bigcap \{E_\delta(L - P), P \in \text{Spec}(L), \mu \subseteq L - P\}$. It remains to prove that $(\mu) \subseteq L - P$ if and only if $\mu \not\in P$.

Hence $E_T(\{\mu\}) = \bigcap \{E_\delta(L - P), P \in \text{Spec}(L), \mu \not\in P\}$.

In Example 4.1, we show that the inclusion is proper. This motivates the following definition.

**Definition 4.1.** A filter $T$ of $L$ is said to be a strong fixed filter relative to an ideal $I$ of $L$ if $T = E_T(\kappa)$.

**Example 4.2.** Consider the $M$-$S$-algebra in Example 3.1. Suppose that $T = [\delta] = \{\delta, 1\}$ and $\kappa = \{\mu, o\}$, thus $E_T(\kappa) = T$. Hence $T$ is a strong fixed filter relative to $\kappa$. Take $R = \{\mu, \delta, 1\}$. We have $E_R(\kappa) = L$. Then $R$ is not a strong fixed filter relative to $\kappa$.

**Proposition 4.1.** Every strong fixed filter relative to an ideal $\kappa$ of $L$ is a fixed filter of $L$ relative to $\kappa$.

Proof. Assume that $T$ is a strong fixed filter relative to an ideal $\kappa$ of $L$. Then $E_T(\kappa) = T$. Since $T \subseteq E_T(\kappa) \subseteq E_T(\kappa) = T$, then $T = E_T(\kappa)$.

**Proposition 4.2.** If $\kappa_1, \kappa_2 \in \mathcal{I}(L)$ and $T$ is a strong fixed filter of $L$ relative to $\kappa_2$ satisfying $\kappa_1 \subseteq \kappa_2$. Then $T$ is a strong fixed filter of $L$ relative to $\kappa_1$.

Proof. Suppose that $T$ is a strong fixed filter of $L$ relative to an ideal $\kappa_2$. Then $T = E_T(\kappa_2)$. We have $\kappa_1 \subseteq \kappa_2$. Therefore $E_T(\kappa_1) \subseteq E_T(\kappa_2) = T$. Also, $T \subseteq E_T(\kappa_1)$. Hence $T = E_T(\kappa_1)$.

**Remark 4.1.** A prime filter $P$ of $L$ is not necessarily a strong fixed filter of $L$ relative to the ideal $L - P$. Consider the following Hasse diagram $L$ in Figure 2. Define a unary operation $^\circ$ on $L$ by $\lambda^\circ = \eta^\circ = \eta$, $\delta^\circ = \nu^\circ = \delta$, $1^\circ = \gamma^\circ = \beta^\circ = \rho^\circ = 0$, $0^\circ = 1$. Then $(L, o) \in M$-$S$. Take $P = \{\eta\} = \{\eta, \rho, 1\}$. We have $L - P = \{0, \nu, \delta, \lambda, \beta, \gamma\}$. Then,

$$E_P(L - P) = \left\{ u \in L : u \lor \gamma^\circ \in P \text{ for some } \gamma \in L - P \right\}$$

$$= \left\{ u \in L : u \lor 0 \in P, \text{ or } u \lor \delta \in P, \text{ or } u \lor \eta \in P, \text{ or } u \lor 1 \in P \right\}$$

$$= L \not\subseteq P.$$

Figure 2: $L$. 

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Proposition 4.3. If $M$ is a maximal filter of $\mathcal{L}$, then $E_M(\mathcal{L} - M) = \mathcal{L}$ or $M$ is a strong fixed filter of $\mathcal{L}$ relative to the ideal $\mathcal{L} - M$.

Proof. We have proved that $E_M(\mathcal{L} - M)$ is a filter which contains $M$. Then, either $E_M(\mathcal{L} - M) = M$ or $E_M(\mathcal{L} - M) = \mathcal{L}$. \hfill $\square$

Now, we study the lattice structure of the following sets

$$E(\kappa) = \{ E_T(\kappa), \ T \in \mathfrak{F}(\mathcal{L}) \},$$

$$\overline{E_T} = \{ E_T(\kappa), \ \kappa \in \mathfrak{I}(\mathcal{L}) \}.$$

Theorem 4.3. Let $\kappa$ be a proper ideal of an MS-algebra $\mathcal{L}$. Then $(E(\kappa); \lor, \cap, E(1), E(\mathcal{L}))$ is a bounded distributive lattice by defining

$$E_T(\kappa) \lor E_R(\kappa) = E_{T \lor R}(\kappa),$$

$$E_T(\kappa) \cap E_R(\kappa) = E_{T \cap R}(\kappa).$$

Moreover, if $\mathcal{L}$ is a complete lattice, then $E(\kappa)$ is complete.

Proof. The element $E(1)$ is the smallest in $E(\kappa)$. We have that $\{1\} \subseteq T$ for every $T \in \mathfrak{F}(\mathcal{L})$. This implies that $E(1) \subseteq E_T(\kappa)$. Also, $E(\kappa)$ is the largest element in $E(\kappa)$, since $T \subseteq \mathcal{L}$ for every $T \in \mathfrak{F}(\mathcal{L})$, then $E_T(\kappa) \subseteq E(\kappa)$. In fact, $E(\kappa) = \mathcal{L}$ since for every $\alpha \in \mathcal{L}$, $\alpha = 0^0 \in \mathcal{L}$.

By Lemma 4.1, $E_T(\kappa) \cap E_R(\kappa) = E_{T \cap R}(\kappa)$. It remains to prove that $E_T(\kappa) \lor E_R(\kappa) = E_{T \lor R}(\kappa)$. We have that $f, g \leq f \lor g$ for every $f \in T$ and $g \in R$. Then $f \lor g \in T, R$. Thus $T \lor R \subseteq T, R$. This implies that $E_{T \lor R}(\kappa) \subseteq E_T(\kappa), E_R(\kappa)$. It follows that $E_{T \lor R}(\kappa) \subseteq E_T(\kappa) \lor E_R(\kappa)$. On the other hand, let $z \in E_T(\kappa) \lor E_R(\kappa)$. Then $z = \lambda \lor \mu$ for some $\lambda \in E_T(\kappa)$ and $\mu \in E_R(\kappa)$. Then $\lambda \lor a^0 \in T$ and $\mu \lor b^0 \in R$ for some $a \in \kappa$ and $b \in \kappa$. Hence,

$$z \lor (a \lor b)^0 = (\lambda \lor \mu) \lor (a \lor b)^0 = (\lambda \lor a^0) \lor (\mu \lor b^0) \in T \lor R.$$

We conclude that $z \in E_{T \lor R}(\kappa)$. Then $E_T(\kappa) \lor E_R(\kappa) \subseteq E_{T \lor R}(\kappa)$. The completeness of $E(\kappa)$ is immediate. \hfill $\square$

Theorem 4.4. If $J$ and $K$ are ideals of an MS-algebra $\mathcal{L}$ and $E_T(J) \lor E_T(K) \subseteq E_T(J \lor K)$, then $(E_T; \lor, \cap, E_T(\{0\}), E_T(\mathcal{L}))$ is a bounded distributive lattice.

Proof. The element $E_T(\{0\})$ is the smallest. Since $\{0\} \subseteq J$ for every $J \in \mathfrak{I}(\mathcal{L})$, then $E_T(\{0\}) \subseteq E_T(J)$. Also, $E_T(\mathcal{L})$ is the largest element in $E_T$. Since $J \subseteq \mathcal{L}$ for every $J \in \mathfrak{I}(\mathcal{L})$, then $E_T(J) \subseteq E_T(\mathcal{L})$.

From Lemma 4.1, $E_T(J) \cap E_T(K) = E_T(J \cap K)$. It remains to prove that $E_T(J \lor K) \subseteq E_T(J \lor K)$. Since $J \lor K \subseteq J, K$, then $E_T(J \lor K) \subseteq E_T(J), E_T(K)$, thus $E_T(J \lor K) \subseteq E_T(J) \lor E_T(K)$. By assumption, $E_T(J \lor K) \subseteq E_T(J \lor K)$. Hence, $(E_T; \lor, \cap, E_T(\{0\}), E_T(\mathcal{L}))$ is a bounded distributive lattice. \hfill $\square$
5 Conclusions and Future work

In this paper, a new definition is presented and notated by $E_T(Z)$. We proved that $E_T(Z)$ is a filter containing $T$, consequently $E_T(Z)$ is called an extended filter of $T$. We concerned in studying a special type of extended filters called fixed filters. In fact, a fixed filter $T$ is the smallest possible extended filter containing $T$ with respect to a set.

A generalisation of $E_T(Z)$ was introduced by defining the strong extensions donated by $E_T(Z)$. The extension $E_T(Z)$ contains both $T$ and $E_T(Z)$. We proved by a counter example that both $E_T(Z)$ and $E_T(Z)$ are not the same. In future work, we may study the homomorphisms and topological spaces related to of $E_T(Z)$. Also, we can study the fuzzification of $E_T(Z)$.

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References


