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Dynamical Analysis of Stochastic Predator-prey Model with Scavenger

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Abstract

In this paper, we studied the dynamic properties of predator-prey and scavenger three species system by using ergodic invariant measures. Pengyu Ma. find the five points of dynamical bifurcation of the stochastic model, which happened between extinction and survival of each species. Environmental noise was added and proved by the fact that driving force produced by environmental noise influence the system and it was find that system may extinct or partially extinct. Here, we have analysed the stochastic bifurcation phenomena of the prey-predator with scavenger system from the nature of dynamic bifurcation. The phase plots and time diagram plotted for the different values of parameters. We have verified all the results by numerical simulations.

Keywords: stochastic dynamical bifurcation; invariant sets; stochastic process; Lyapunov exponents; ergodic invariant probability measures.

1 Introduction

Many researchers developed Mathematical ecosystem models and studied the growth and decay of the species in a confined region. First Mathematical biological model was studied by Lokta [21] in 1925. After one year an other Mathematical biology model was introduced by Voltera in 1926. After that many scientists and researchers worked on Mathematical biological and ecological models [16, 17, 19], Wilson et al. [33] discovered a predator-prey model on Lion, zebra and cheetah, impala. The model Dynamical analysis of Two-Preys and One Predator Interaction Model with an Allee Effect on Predator [20] was developed on two prey species and one predator and they studied how the population was affected by the used parameters. Gupta et al. [10] gave the model on Scavenger, predator and prey with quadratic harvesting. The dynamical properties were also analysed in model. Satar and Naji [30] gave the stability of predator-prey-scavenger model with michaelis-menten type of harvesting function. In [6] Stochastic prey predator model with additional food for predator was studied and analysed the behavior of the model.

Previte et al. [24] analyzed Period doubling cascades in a predator-prey model with a scavenger. In [2] author discussed the dynamics of a three species ratio dependent of food chain model with intra specific competition with in the top predator. In other references [25, 15] Chaotic measurement were also analysed. Rich Dynamics of a Predator-Prey system with different kinds of functional responses [29] was studied for the various parameters.

Khajanchi et al. [18, 4] studied brain tumor and fuzzy HTLV-I infection model. The brain tumor model analysed with Immune System interactions. In research paper [27] analysed the allee effect relevant to stochastic cancer model. The researcher determined the very realistic results in model. Sengupta et al. [31] explored the Stochastic non-autonomous holling type-III prey-predator model with predator intra-specific competition.

In references [26, 6] stochastic thermodynamical and ecological model were studied. Samanta et al. [23] analysed the dynamics of an additional food provided predator-prey system with prey refuge dependent on both species and constant harvest in predator. An interesting model was developed by Khajanchi et al. [28] on the effect of fear on growth rate of prey species. In model researchers find that fearing of predator affect the population growth of prey.

Some researchers analysed that the relation between populations and the noise provided by environment can improve or reduce the populations which already exist in system. So that the system may coexist or partially extinct or extinct. In this paper we studied and tried to give answers for these questions. Here the predator-prey model with scavenger has the following form;

$$\begin{aligned} dx &= x\big(r(1-x/K) - ay - bz - c\alpha x - dx^2\big)dt, \\ dy &= y\big(eax - fy - g\alpha - hy\big)dt, \\ dz &= z\big(jbx + ky - l - m\alpha - nz\big)dt, \end{aligned} \tag{1}$$

where x is prey and y is predator and z is scavenger. In the system, parameter r define the natural growth rate of prey species, K the carrying capacity without predation harvesting and toxicant. α is combined harvesting effort. a, b and k are predation and scavenge with positive maximum attack rate. c, g and m are positive catch ability coefficients. d, h and n are coefficient of toxicity of prey, predator and scavenger respectively. f and l are predator and scavenger decay with natural death rates respectively. e and j are conversion rates of prey to predator and scavenger.

2 Stochastic Form of System (1)

Many multi-species problems discussed without any biotic effect. But some biotic effect can also affect the nature of the problem. Due to biotic effect, in some models, population may grow very fast or go extinct. Mao et al. [22] the population dynamics analysed by adding environmental brownian noise. Deng et al. [7] also discussed that white noise directly can affect the population of the species and growth or decay of the population may be exponentially. The principle of competitive exclusion for a stochastic Lotka-Voltera model with super and mid level predators competing for one prey is studied by Cao et al. [5], Gakkhar et al. [9] revealed dynamical behaviour of super and mid level predators over a single prey. In other problem, Sun et al. [32] studied the dynamical behavior of a stochastic two species monod competition chemostat model.

Fluctuation in the environment may cause a extent in model. We are studying the phenomenon by using following stochastic predator-prey with scavenger model

$$\begin{aligned} dx &= \big(rx(1 - x/K) - axy - bxz - c\alpha x - dx^3\big)dt + \beta_1 x dB_1(t), \\ dy &= \big(eaxy - fy - g\alpha y - hy^2\big)dt + \beta_2 y dB_2(t), \\ dz &= \big(jbxz + kyz - lz - m\alpha z - nz^2\big)dt + \beta_3 z dB_3(t). \end{aligned} \tag{2}$$

System (2) is developed for scavenger, mid level predator and prey species. In the system (2), B_1, B_2 and B_3 stand for independent standard brownian motions; β_1, β_2, and β_3 are the intensities of white noises. In our model we considered that the environmental change, mainly affect the growth and death rate of the population. However, due to environmental effect the population may go extinct. In our model we find, how the random factors influence the model? These things show that noise, we are introducing can affect the system, and from these we also find how does this white noise affect the mainly stochastic model (2.1). Mao et al. [22] described the population model in which model they studied how the brownian noise effect the population and change into the nature of the system. Some other stochastic population models [12, 14] developed and studied by many researchers. In model we analysed, how environmental noise can improve the biological populations or may decrease the populations or may be extinct the populations or it may be coexist.

3 Important Invariant Sets for System (2)

In this model we will study how environmental noise affect the population and how it's differ from the biological population without noise. To study the dynamic properties of the population model, we are using Lyapunov Characteristic Exponent of invariant measures on invariant sets.

Theorem 3.1. *The solution process,*

$$\mathcal{Z} \triangleq (x(t), y(t), z(t)),$$

of the stochastic system (2) of the population model is regular with initial value $(x(0), y(0), z(0))$. The initial value lie in the invariant set S, where $S \triangleq \{(x, y, z) \mid x > 0, y > 0, z > 0\}$.

Proof. The similar proof has been given in reference [37]. So that we are omitting the Proof. □

Remark 3.1. *By using Theorem 3.1, S is a invariant set, which is three-dimensional. We get the following;*

$$S_1 = \{(x, y, z) \mid x > 0, \quad y > 0, \quad z = 0\},$$
$$S_2 = \{(x, y, z) \mid x > 0, \quad y = 0, \quad z > 0\},$$
$$S_3 = \{(x, y, z) \mid x = 0, \quad y > 0, \quad z > 0\},$$

invarient subsets. It is define that $x(t) \equiv 0$ when x(0)=0, $y(t) \equiv 0$ when y(0)=0 and $z(t) \equiv 0$ when z(0)=0. Hence, following invariant subsets are

$$S^1 = \{(x, y, z) \mid x > 0, \quad y = 0, \quad z = 0\},$$
$$S^2 = \{(x, y, z) \mid x = 0, \quad y > 0, \quad z = 0\},$$
$$S^3 = \{(x, y, z) \mid x = 0, \quad y = 0, \quad z > 0\},$$

and origin is a sub-invariant set for the given stochastic system (2). Hence, the system has S_1, S_2, S_3, S^1, S^2, S^3, S and $(0, 0, 0)$ invariant sub-sets for the invariant set $\mathcal{S} \triangleq \{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0\}$. By study we find that the sub-invariant sets S_3, S^2 and S^3 are meaningless.

Theorem 3.2. *We can easily prove, the sets S_3, S^2, S^3 are meaningless. As we know no predator and scavenger will survive without prey.*

Proof. We are using Theorems 1.1 and 1.3, given in reference [12] to study stochastic system (2). Here, we also used Theorem 1.4 that given in reference [12]. We take the system (2) on the 2-dimensional boundary set $S_4 = \{(x, y, z) \mid x = 0, y \geq 0, z \geq 0\}$. Let $\mu(.)$ be the dirac measure at (0,0,0), hence, the measure $\mu(.)$ is a probabilistic ergodic invariant. The first and second Lyapunov exponent of measure $\mu(.)$ are computed as,

$$v_1(\mu) = \int_{(0,0,0)} \left((r(1 - x/K) - ay - bz - c\alpha - dx^2) - \frac{\beta_1^2}{2} \right) \mu(dxdydz) = r - c\alpha - \frac{\beta_1^2}{2},$$
$$v_2(\mu) = \int_{(0,0,0)} \left(eax - f - g\alpha y - hy - \frac{\beta_2^2}{2} \right) \mu(dxdydz) = -f - g\alpha - \frac{\beta_2^2}{2},$$
$$v_3(\mu) = \int_{(0,0,0)} \left(jbx + ky - l - m\alpha - nz - \frac{\beta_2^2}{2} \right) \mu(dxdydz) = -l - m\alpha - \frac{\beta_3^2}{2},$$

where $v_1(\mu)$ and $v_2(\mu)$ are first and second Lyapunov exponents of measure μ. Since $v_1(\mu)$ and $v_2(\mu)$ both are less than zero therefore $\mu(.)$ only measure on S_4. It gives us the predator and scavenger will go extinct without prey. Hence S_3 is useless invariant set similarly we can show that the invariant sets S^2 and S^3 are not useful. □

Corollary 3.1. *For any real $q > 0$, and initial value $\mathcal{Z}(0) \in S$, the stochastic process $\mathcal{Z}(t)$ of system (2) satisfied,*

$$\lim_{t \to \infty} \sup E \mid \mathcal{Z}(t) \mid^q \leq C_1,$$

where C_1 is a constant which must be positive.

4 LCE for Ergodic Measures on Invariant Subsets

Here, we are considering the system (2) on the first-dimensional boundary $S_0^1 = \{(x, y, z) \mid x \geq 0, y = 0, z = 0\}$ or

$$dx = (rx(1 - x/K) - c\alpha x - dx^3)dt + \beta_1 x dB_1(t). \tag{3}$$

Corollary 4.1. *The reference [14], for system (4.1) provides* $\lim_{t\to\infty} x(t) = 0$ *when* $\beta_1^2 \geq 2r$ *and from the reference [8], we can write the system has a unique distribution* $\mu_1(.)$ *with density function* $f(x)$ *when* $\beta_1^2 \leq 2r$*, where,*

$$f(x) = \frac{\beta^\kappa x^{\kappa-1} e^{-\beta x}}{\Gamma\kappa}, \qquad x > 0, \qquad \beta > 0, \qquad \kappa > 0.$$

Where $\beta = \dfrac{2r}{K\beta_1^2}$ *and* $\kappa = \dfrac{2(r - c\alpha)}{\beta_1^2} - 1$*. For the system (2), if* $v_1(\mu) = r - c\alpha - \dfrac{\beta_1^2}{2} < 0$*, therefore, the Lyapunov exponent of invariant measure* $\mu(.)$ *is less than zero, hence there is a invariant measure* $\mu(.)$ *on* $\mathcal{S}$*. For* $v_1(\mu) > 0$*, there is an other invariant measure* $\mu_1(.)$ *on* S^1*. We know that the measure* $\mu(.)$ *is not stable and measure* $\mu_1(.)$ *can be defined as a boundary measure on* $\delta\mathcal{S}$*. Here, we are not making any difference between them and the difference is depend on the conditions.*

Theorem 4.1. *For the boundary measure*$\mu_1(.)$*, the Lyapunov exponents are* $v_1(\mu_1) = 0$*.*

$$v_2(\mu_1) = -f - g\alpha - \frac{\beta_2^2}{2} + ae\int_0^\infty x\frac{\beta^\kappa x^{\kappa-1}e^{-\beta x}}{\Gamma\kappa}\,dx,$$
$$v_3(\mu_1) = -l - m\alpha - \frac{\beta_3^2}{2} + jb\int_0^\infty x\frac{\beta^\kappa x^{\kappa-1}e^{-\beta x}}{\Gamma\kappa}\,dx.$$

Proof. From the reference [12], the Lemma 2.1 shows $v_1(\mu_1) = 0$. The rest measures $v_2(\mu_1)$ and $v_3(\mu_1)$ can be calculate easily. So we are omitting proof. □

Let $v_1(\mu) > 0$ and $v_2(\mu_1) > 0$. Then there is probability measure which is ergodic invariant μ_{12} on S_1. Therefore the 2-dimensional boundary system will become,

$$\begin{aligned} dx &= (rx(1 - x/K) - axy - bxz - c\alpha x - dx^3)dt + \beta_1 x dB_1(t), \\ dy &= (eaxy - fy - g\alpha y - hy^2)dt + \beta_2 y dB_2(t). \end{aligned} \tag{4}$$

From reference [12], Lemma 2.1 shows $v_1(\mu_{12}) = 0$ and $v_2(\mu_{12}) = 0$, For the measure μ_{12} the Lyapunov exponent can be computed as,

$$v_3(\mu_{12}) = -l - m\alpha - \frac{\beta_3^2}{2} + \int_{S_1}(jbx + ky)\mu_{12}\,(dx, dy).$$

In same manner we have an measure μ_{13} on S_2, which is ergodic. When $v_1(\mu) > 0$ and $v_3(\mu_1) > 0$, then the 2-dimensional boundary system will become

$$\begin{aligned} dx &= (rx(1 - x/K) - bxz - c\alpha x - dx^3)dt + \beta_1 x dB_1(t), \\ dz &= (jbxz + -lz - m\alpha z - nz^2)dt + \beta_3 z dB_3(t). \end{aligned} \tag{5}$$

Here, we have $v_1(\mu_{13} = 0)$ and $v_3(\mu_{13}) = 0$ and

$$v_2(\mu_{13}) = -f - g\alpha - \frac{\beta_2^2}{2} + ae\int_{S_2}(xae)\mu_{13}\,(dx, dz).$$

5 Stochastic Extinction and Persistence

The basic definition of the strong stochastic persistence has been defined from the reference [36, 12] for system (2) as we can see below.

Definition 5.1. *Let $\mathcal{Z}(t)$ be stochastic process and the stochastic process will be persist if transition probability $\mathcal{Z}(t)$ converge to $\nu(.)$ and the process has an invariant probability measure $\nu(.)$. We are denoting*

$$\Psi_t(.) = \frac{1}{t}\int_0^t 1_{\{z(t)\epsilon\}} ds, \qquad t > 0,$$

as a normalized random measure for the stochastic process $\mathcal{Z}(t)$. Suppose,

$$\mathbb{M} = \{\mu(.), \mu_1(.), \mu_{12}(.), \mu_{13}(.)\},$$

where the elements of $\mathbb{M}$ are ergodic invariant probability measures of $\mathcal{Z}(t)$. For the system (2), reference [12] provide the following stability theorem.

Theorem 5.1. *For any η belong into $\mathbb{M}$ we have the following;*

(*i*) *If $\max_{i=1,2,3} v_i(\eta) > 0$, then the measure $\eta(.)$ is unstable.*

(*ii*) *If $\max_{i=1,2,3}\{v_i(\mu)\} < 0$ then the dirac measure $\mu(.)$ is stable.*

(*iii*) *If $\mu_1(.)$, if $\max_{i=2,3} v_i(\mu_1) < 0$, then one-dimensional measure μ_1 is stable.*

(*iv*) *If $v_3(\mu_{12}) < 0$ and $\max_{i=1,2}\{v_i(\mu)\} > 0$ then two-dimensional measure μ_{12} is stable.*

(*v*) *If $v_2(\mu_{13}) < 0$ and $\max_{i=1,3}\{v_i(\mu)\} > 0$ then two-dimensional measure $\mu_{13}(.)$ is stable.*

For stability of the measure $\eta(.)$. the limit of $\Psi_t(.)$ is $\{\eta(.)\}$ definitely and

$$\lim_{t\to\infty} \frac{\log \mathcal{Z}_i(t)}{t} = v_i(\eta).$$

Theorem 5.2. *Let $v_1(\mu) < 0$. Therefore all the species $x(t)$, $y(t)$ and $z(t)$ definitely will converge to zero, it shows that the system (2) will be extinct.*

Proof. We have $v_2(\mu) = -f - g\alpha - \dfrac{\beta_2^2}{2} < 0$ and $v_3(\mu) = -l - m\alpha - \dfrac{\beta_3^2}{2} < 0$, then $\max_{i=1,2,3}\{v_i(\mu)\} < 0$ when $v_1(\mu) < 0$, this gives the measure $\mu(.)$ is stable and

$$\lim_{t\to\infty} \frac{\log \mathcal{Z}_i(t)}{t} = v_i(\mu) < 0.$$

Here the values of i are 1, 2 and 3 respectively. Therefore, the species $x(t)$, $y(t)$ and $z(t)$ definitely will converges to zero and the system (2) will be extinct. □

Theorem 5.3. *Let $v_1(\mu) > 0$. Therefore the dynamical properties for the system are obtained by $v_2(\mu_1)$ and $v_3(\mu_1)$. Hence we can divide these properties it into three cases:*

(*i*) *For $v_2(\mu_1) < 0$ and $v_3(\mu_1) < 0$, The species y and z will be extinct.*

(*ii*) *For $v_2(\mu_1) > 0$ and $v_3(\mu_1) < 0$, the species z will extinct.*

(*iii*) *For $v_2(\mu_1) < 0$ and $v_3(\mu_1) > 0$, the species y will extinct.*

Proof. From reference [37] the results easily can be verified, hence we are omitting the proof. □

Theorem 5.4. *If $v_1(\mu) > 0$, $v_2(\mu_1) > 0$ and $v_3(\mu_1) > 0$, then there are following four cases arise;*

(*i*) *For* $v_2(\mu_{13})$ *and* $v_3(\mu_{12})$ *are less than zero, the randomized measure converges to measure* $\mu_{12}(.)$ *and* μ_{13} *with probability* p_1 *and* p_2 *respectively. Here the convergence defined for randomized measure is weak convergence. Suppose,*

$$p_1 = \mathbb{P}\{\Psi(.) \to \mu_{12}(.)\},$$
$$p_2 = \mathbb{P}\{\Psi(.) \to \mu_{13}(.)\},$$

where $p_1, p_2 > 0$ *and* $p_1 + p_2 = 1$.

(*ii*) *For* $v_2(\mu_{13})$ *is greater than zero and* $v_3(\mu_{12})$ *is less than zero, the species* $z(t)$ *definitely converges to zero and the randomized measure converges to the measure* $\mu_{12}(.)$. *Here, the convergence is the weak convergence.*

(*iii*) *For* $v_2(\mu_{13})$ *is less than zero and* $v_3(\mu_{12})$ *is greater than zero, the species* $y(t)$ *definitely will converge to zero and the randomized measure converges to measure* $\mu_{13}(.)$.

(*iv*) *For* $v_2(\mu_{13})$ *and* $v_3(\mu_{12})$ *are greater than zero, so that there will be a invariant probability measure* $\nu(.)$ *on* S.

Proof. It is given that $\max_{i=1,2,3}\{v_i(\mu)\} > 0$ and $v_1(\mu) > 0$, therefore the dirac measure $\mu(.)$ is not stable. In same manner $\max_{i=1,2,3}\{v_i(\mu_1)\} > 0$, since $v_2(\mu_1)$ and $v_3(\mu_1)$ are greater than zero, then measure $\mu_1(.)$ is not stable.

(i) If $v_2(\mu_{13}) < 0$ and $v_3(\mu_{12}) < 0$, then results we find from Theorem 1.3 in reference [12]. Since $\mathbb{M}^1 = \{\mu_{12}(.), \mu_{13}(.)\}$.

(ii) If $v_2(\mu_{13}) > 0$ and $v_3(\mu_{12}) < 0$, then $\max_{i=1,2,3}\{v_i(\mu_{13})\} > 0$. Hence measure $\mu_{13}(.)$ is not stable and measure $\mu_{12}(.)$ is stable and

$$\lim_{t\to\infty} \frac{\log \mathcal{Z}_3(t)}{t} = v_3(\mu_{12}) < 0.$$

This implies the randomized measure converges to measure $\mu_{12}(.)$ and $z(t)$ definitely converges to zero .

(iii) If $v_2(\mu_{13}) > 0$ and $v_3(\mu_{12}) > 0$, we can prove it similarly as when $v_2(\mu_{13}) > 0$ and $v_3(\mu_{12}) < 0$, therefore we are omitting the proof.

(iv) For $v_2(\mu_{13}) > 0$ and $v_3(\mu_{12}) > 0$, a boundary measure $\chi(.)$ on boundary $\delta\mathcal{S}$ is $\chi(.) = k_1\mu(.) + k_2\mu_1(.) + k_3\mu_{12}(.) + k_4\mu_{13}(.)$ with $k_1 + k_2 + k_3 + k_4 = 1$ and $k_1, k_2, k_3, k_3 \geq 0$. It is the linear combination of $\mu(.)$, $\mu_1(.)$, $\mu_{12}(.)$ and $\mu_{13}(.)$.

This simply prove that maximum value of $\{v_i(\chi)\}$ for all $i = 1, 2, 3$ is greater than zero for $\chi(.)$. So that any measure on $\delta\mathcal{S}$ is not stable and it produce a invariant measure $\nu(.)$. Therefore the system (2) is stochastically tenacious. □

6 Biological Importance and Stochastic Bifurcations for Bifurcation Points

From reference [38] we find that there are dynamical and phenomenon approach for the research in stochastic bifurcations. Phenomenon approach measure the qualitative change. Through the dynamical approach we study the stability of invariant measure, it is explained in reference [3]. Here, in this paper we are using dynamical approach to study the stochastic system (2).

Theorem 6.1. *$v_1(\mu)$,$v_2(\mu_1)$, $v_3(\mu_1)$, $v_2(\mu_{13})$ and $v_3(\mu_{12})$ are dynamical bifurcation point for system (2).*

Proof. Here, we are giving importance and explanation of these dynamical bifurcation point;

(i) $v_1(\mu)$. Stability of measure $\mu(.)$ defined from value of $v_1(\mu)$, If $v_1(\mu) > 0$ the measure will be stable. However, If $v_1(\mu) < 0$, then measure will not be stable and another new measure come into effect.

(ii) $v_2(\mu_1)$. For $v_1(\mu) > 0$ and $v_3(\mu_1) < 0$. The stability of measure $\mu_1(.)$ can be define from the nature of $v_2(\mu_1)$. If $v_2(\mu_1) < 0$, the measure $\mu_1(.)$ will be stable. However, if $v_2(\mu_1) < 0$ then, the measure will be unstable and another new measure come into effect.

(iii) $v_3(\mu_1)$. For $v_1(\mu) > 0$ and $v_2(\mu_1) < 0$. The stability of measure $\mu_1(.)$ find as if $v_3(\mu_1) < 0$ then measure will be stable. However $v_3(\mu_1) > 0$, then measure will be unstable and a new measure $\mu_{13}(.)$ generates by the system.

(iv) $v_3(\mu_{12})$. For $v_1(\mu) > 0$, $v_2(\mu_1) > 0$, $v_3(\mu_1) > 0$ and $v_2(\mu_{13}) > 0$. The boundary measure $\mu_{12}(.)$ will be stable for $v_3(\mu_{12}) < 0$ and for $v_3(\mu_{12}) > 0$, the measure will be unstable and a new measure $\nu(.)$ generates by system which is also stable.

(v) $v_2(\mu_{13})$. For $v_1(\mu) > 0$, $v_2(\mu_1) > 0$, $v_3(\mu_1) > 0$ and $v_3(\mu_{12}) > 0$. The measure $\mu_{13}(.)$ will be stable for $v_2(\mu_{13}) < 0$. However, If $v_2(\mu_{13}) > 0$, the measure $\mu_{13}(.)$ will be unstable and a new invariant measure $\nu(.)$ come into effect which is stable.

We studied and find all five bifurcation point has biological importance.

Extinction and survivability of species

(i) The value of $v_1(\mu)$ tell us about the survival and extinction of the prey for $v_1(\mu) > 0$, prey will survive and for $v_1(\mu) < 0$, the prey will extinct.

(ii) Survivability and extinction of predator y can be determine by using the value of $v_2(\mu_{13})$. If it's positive and not equal to zero then predator y will survive on other way if it's value less than zero the species y will be extinct. However, if $v_2(\mu_{13})$ does not exist then $v_2(\mu_1)$ use to determine the survivability and extinction of predator z.

(iii) Survivability and extinction of scavenger z can be determine by using the value of $v_3(\mu_{12})$. If it's positive and not equal to zero then scavenger z will survive on other way if it's value less than zero the species z will be extinct. However, if $v_2(\mu_{13})$ does not exist then $v_3(\mu_1)$ use to determine the survivability and extinction of species z.

Extinction and survivability of system

(i) $v_1(\mu)$ has important role to describe the system, the system will survive or extinct, we determine from the $v_1(\mu)$. If $v_1(\mu)$ has non zero positive value then at least one species will survive. However, if $v_1(\mu) < 0$, then system will extinct.

(ii) On Other way if $\min\{v_2(\mu_{13}, v_3(\mu_{12}))\}$ has non zero positive value, then at least one species will survive. However, If $\min\{v_2(\mu_{13}, v_3(\mu_{12}))\} < 0$, then system will extinct.

□

7 Finite Difference Technique for Numerical Simulation

We are using technique, the technique used in literature [13] for the numerical simulation, the following system of equations we get,

$$
\begin{aligned}
x_{i+1} &= x_i + \Big[rx_i(1 - x_i/K) - ax_iy_i - bx_iz_i - c\alpha x_i - dx_i^3\Big] \triangle t \\
&\quad + \beta_1 x_i\sqrt{\triangle t}\xi_i + \frac{\beta_1^2}{2}x_i^2(\triangle t\xi_i^2 - \triangle t), \\
y_{i+1} &= y_i + \Big[eax_iy_i - fy_i - g\alpha y_i - hy_i^2\Big] \triangle t \\
&\quad + \beta_2 y_i\sqrt{\triangle t}\eta_i + \frac{\beta_2^2}{2}y_i^2(\triangle t\eta_i^2 - \triangle t), \\
z_{i+1} &= z_i + \Big[jbx_iz_i + ky_iz_i - lz_i - m\alpha z_i - nz_i^2\Big] \triangle t \\
&\quad + \beta_3 z_i\sqrt{\triangle t}\psi_i + \frac{\beta_3^2}{2}z_i^2(\triangle t\psi_i^2 - \triangle t),
\end{aligned} \tag{6}
$$

where, in system (6) ξ_i, η_i and ψ_i are gaussian stochastic variables, whose values are lie in $N(0,1)$. We are verifying the theoretical results by the numerical simulation. From the graphics we can easily understand the system's dynamical properties. We used some literature for numerical simulations and evaluations. We used literature [11, 35] to derive natural mortality rates, conversion coefficients and so on. White noises's intensities derived from literatures [38, 34].

Conditions given in Theorems 4.1, 5.1, 5.2 and 5.3 are verified by some examples. We compute the Lyapunov exponents for ergodic invariant probability measures. We didn't get any idea about the $v_2(\mu_{13})$ and $v_3(\mu_{12})$. We are using finite difference methods for the numerical simulations and compute the Lyapunov exponents for the different measures. We summarized the numerical simulation in following steps;

(i) The number of point N, which are using for simulation.

(ii) Compute the step length for the data which we have taken in step one.

(iii) N_{kj} represents the data's number in a grid.

(iv) Lyapunov exponents for all the measures, computed by the approximation of the density functions.

The Lyapunov exponents for measures $v_2(\mu_{13})$ and $v_3(\mu_{12})$ are computed in following examples;

Example 7.1. *Let* $r = 0.1$, $K = 18$, $a = 0.52$, $b = 0.45$, $k = 0.35$, $c = 0.11$, $g = 0.31$, $m = 0.21$, $\alpha = 0.40$, $f = 0.40$, $l = 0.32$, $e = 0.85$, $j = 0.20$, $d = 0.50$, $h = 1$, $n = .20\beta_1 = 0.95$, $\beta_2 = .87$, $\beta_3 = .75$. *Then we have*

$$v_1(\mu) = r - c\alpha - \frac{\beta_1^2}{2} = -0.39525 < 0.$$

From Theorem 5.2, we find that the system (2) goes extinct. Figure 1 and Figure 2 show the numerical simulations of Example 7.1. Figure 1 shows system (2) goes extinct with time and phase space shown in Figure 2.

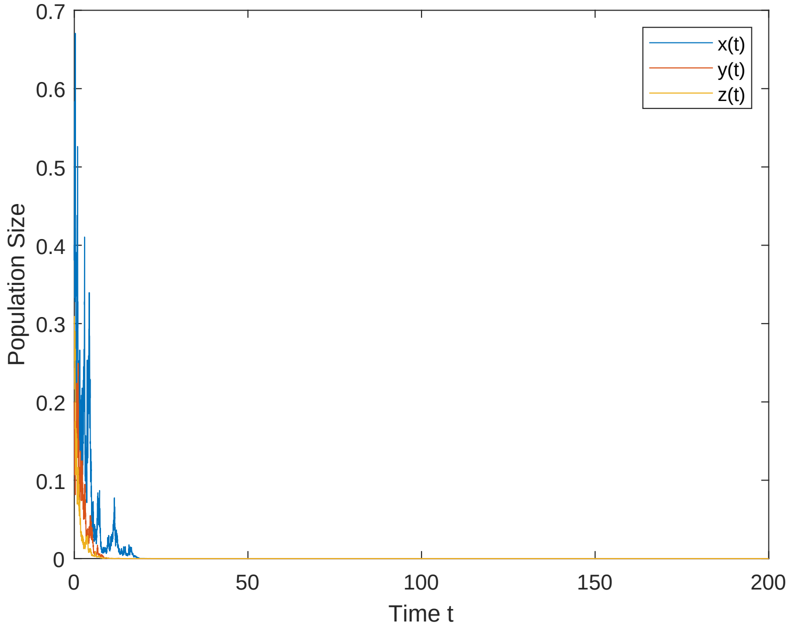

Figure 1: The time plot of Example 7.1 shows the system is extinct.

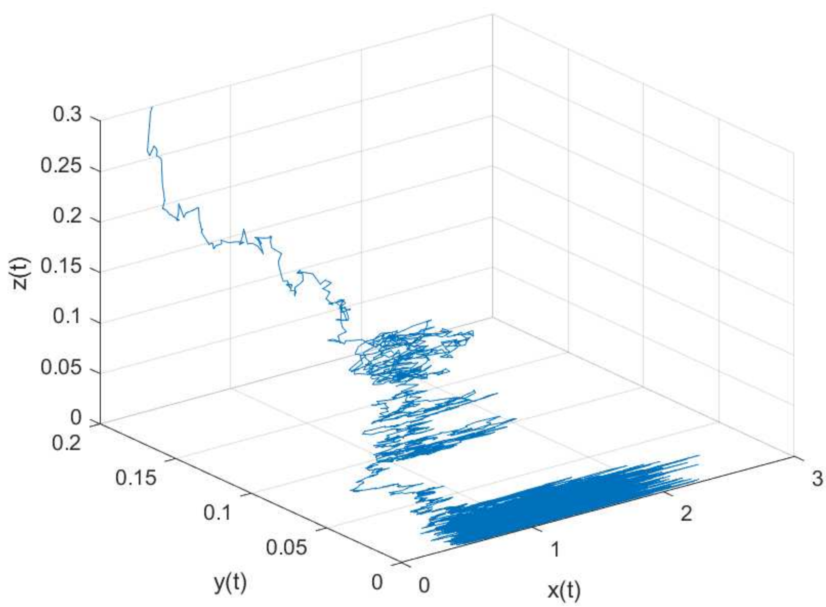

Figure 2: The phase plot of Example 7.1 shows the convergence of measure $\mu(.)$ on $(0,0,0)$.

Example 7.2. *Let* $r = 2$, $K = 18$, $a = 0.52$, $b = 0.45$, $k = 0.35$, $c = 0.11$, $g = 0.31$, $m = 0.21$, $\alpha = 0.40$, $f = 0.40$, $l = 0.32$, $e = 0.85$, $j = 0.20$, $d = 0.50$, $h = 1$, $n = 0.20\beta_1 = 0.95$, $\beta_2 = 0.87$, $\beta_3 = 0.75$. *Then we have*

$$v_1(\mu) = r - c\alpha - \frac{\beta_1^2}{2} = 1.050475 > 0,$$
$$v_2(\mu_1) = -f - g\alpha - \frac{\beta_2^2}{2} + ae\int_0^\infty x\frac{\beta^\kappa x^{\kappa-1}e^{-\beta x}}{\Gamma\kappa}\,dx = -0.979318 < 0,$$
$$v_3(\mu_1) = -l - m\alpha - \frac{\beta_3^2}{2} + jb\int_0^\infty x\frac{\beta^\kappa x^{\kappa-1}e^{-\beta x}}{\Gamma\kappa}\,dx = -0.700901 < 0.$$

By using Theorem 5.3, we find $\Psi(t)$ converges to measure $\mu_1(.)$, and y, z species definitely goes extinct. Figure 3 shows path of the system change with the time t and phase space shown in Figure 4.

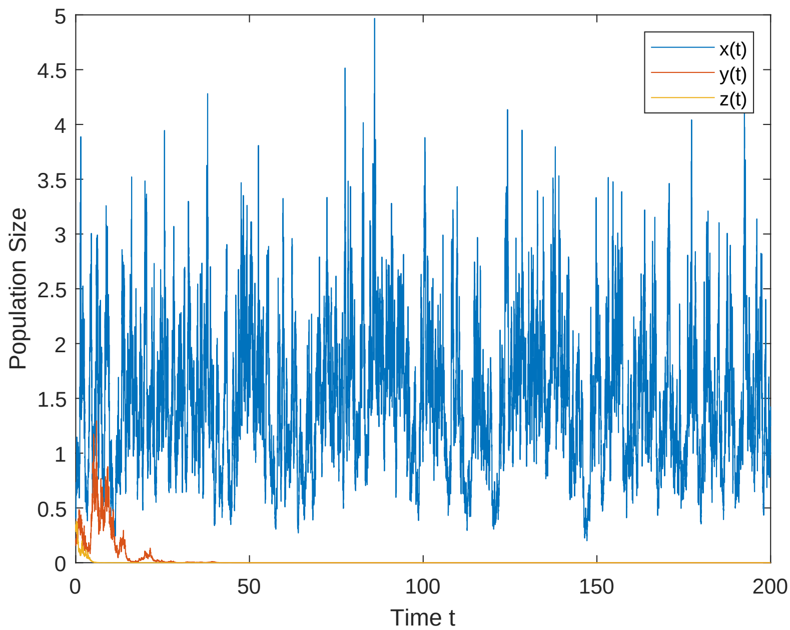

Figure 3: The time plot of Example 7.2 shows species x persist and species y, z are extinct.

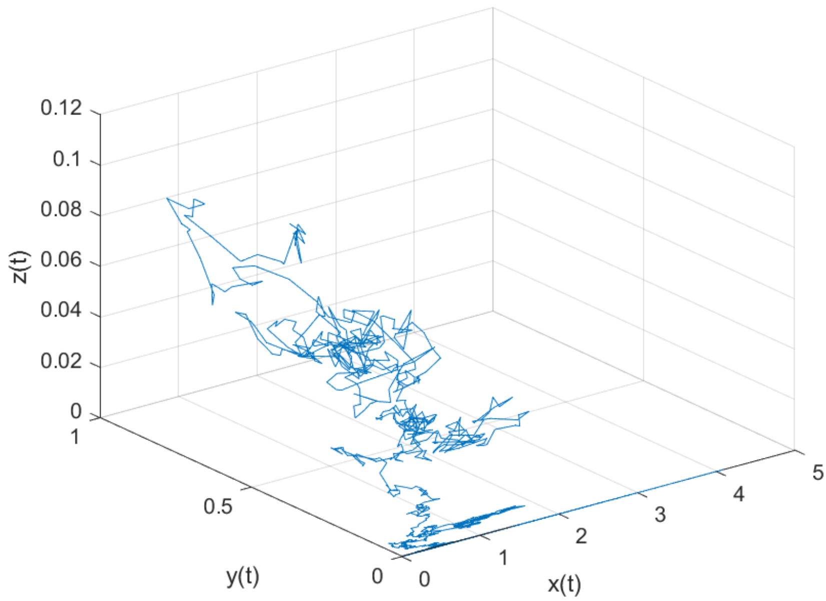

Figure 4: The phase plot of Example 7.2 shows the convergence of $\Psi(t)$ to the measure $\mu_1(.)$.

Example 7.3. *Let* $r = 9$, $K = 18$, $a = 0.52$, $b = 0.45$, $k = 0.35$, $c = 0.11$, $g = 0.31$, $m = 0.21$, $\alpha = 0.40$, $f = 0.40$, $l = 0.32$, $e = 0.85$, $j = 0.20$, $d = 0.50$, $h = 1$, $n = 0.20$, $\beta_1 = .95$, $\beta_2 = 0.07$, $\beta_3 = 0.75$. *Then we have*

$$v_1(\mu) = r - c\alpha - \frac{\beta_1^2}{2} = 8.50475 > 0,$$

$$v_2(\mu_1) = -f - g\alpha - \frac{\beta_2^2}{2} + ae\int_0^\infty x\frac{\beta^\kappa x^{\kappa-1}e^{-\beta x}}{\Gamma\kappa}\,dx = 0.01025 > 0,$$

$$v_3(\mu_1) = -l - m\alpha - \frac{\beta_3^2}{2} + jb\int_0^\infty x\frac{\beta^\kappa x^{\kappa-1}e^{-\beta x}}{\Gamma\kappa}\,dx = -0.68476 < 0.$$

By using Theorem 5.3, we find that $\Psi(t)$ converges to $\mu_{12}(.)$, and species z definitely goes extinct. Figure 5 shows system's path change with the time t and Figure 6 shows the phase space.

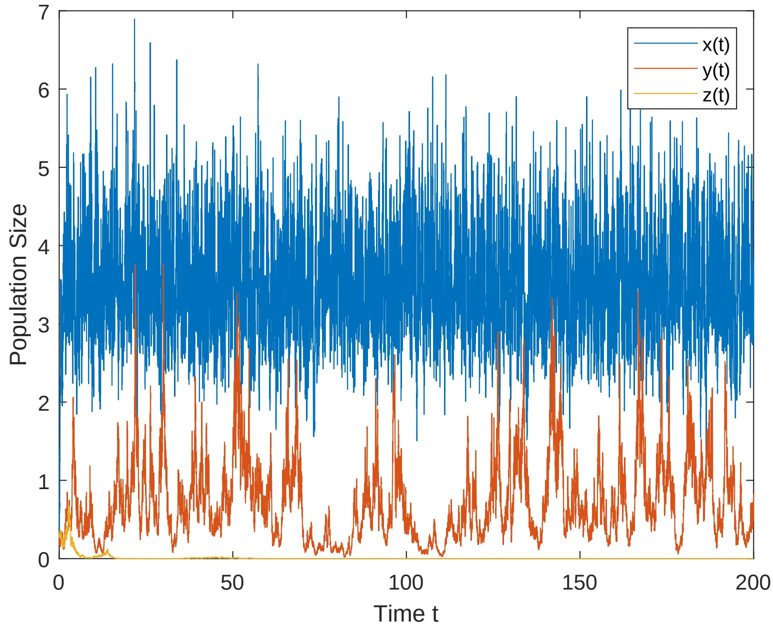

Figure 5: The time plot of Example 7.3 shows the species x and y are persist and species z is extinct.

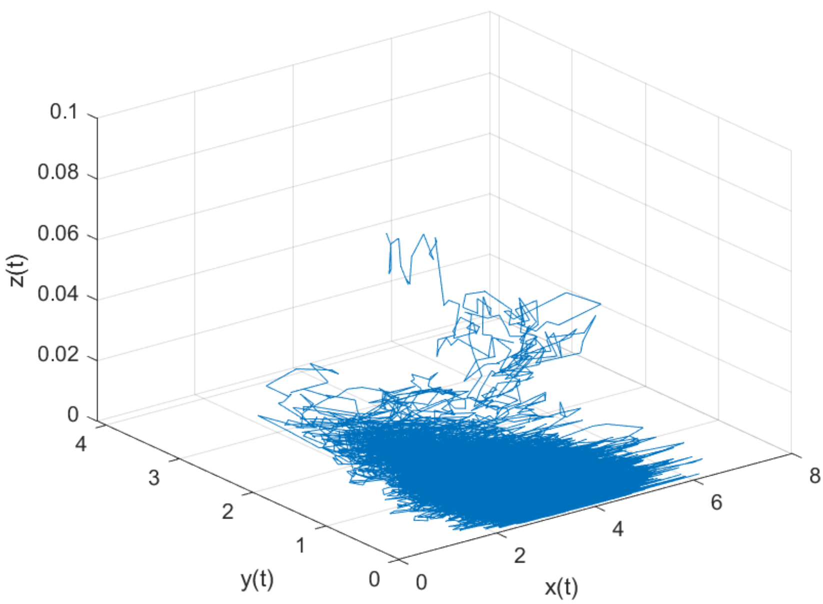

Figure 6: The phase plot of the Example 7.3 shows the convergence of $\Psi(t)$ to measure $\mu_{12}(.)$ on S_1.

Example 7.4. *Let* $r = 10$, $K = 18$, $a = 0.12$, $b = 0.99$, $k = 0.35$, $c = 0.11$, $g = 0.31$, $m = 0.21$, $\alpha = 0.40$, $f = 0.40$, $l = 0.32$, $e = 0.85$, $j = 0.20$, $d = 0.30$, $h = .99$, $n = .20$, $\beta_1 = .65$, $\beta_2 = 0.77$, $\beta_3 = 0.05$. *Then we have*

$$v_1(\mu) = r - c\alpha - \frac{\beta_1^2}{2} = 9.74475 > 0,$$
$$v_2(\mu_1) = -f - g\alpha - \frac{\beta_2^2}{2} + ae\int_0^\infty x\frac{\beta^\kappa x^{\kappa-1}e^{-\beta x}}{\Gamma\kappa}\,dx = -0.816923 < 0,$$
$$v_3(\mu_1) = -l - m\alpha - \frac{\beta_3^2}{2} + jb\int_0^\infty x\frac{\beta^\kappa x^{\kappa-1}e^{-\beta x}}{\Gamma\kappa}\,dx = 0.11304 > 0.$$

By using Theorem 5.3, we find that $\Psi(t)$ converges to μ_{13}, and species y definitely goes extinct. Figure 7 shows system's path change with the time t and Figure 8 shows the phase space.

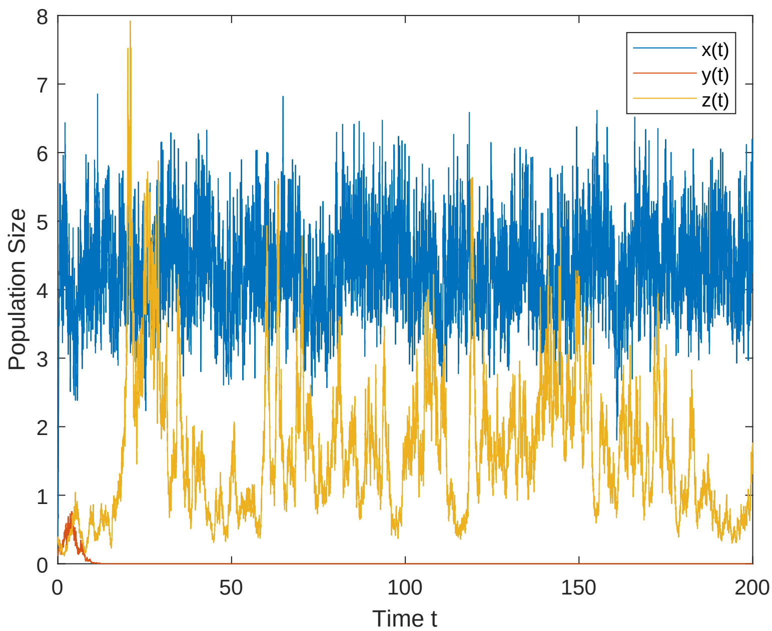

Figure 7: The time plot of Example 7.4 shows x and z are persist and y is extinct.

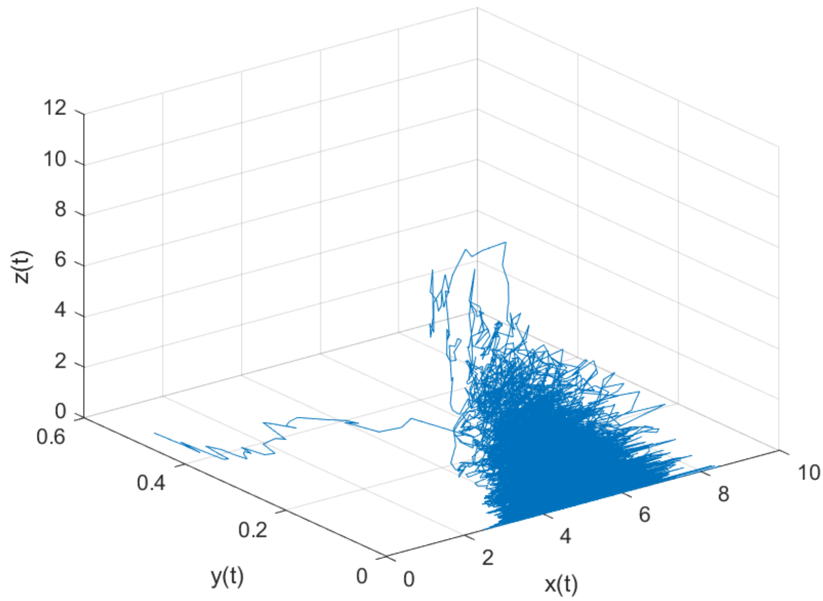

Figure 8: The phase plot of Example 7.4,shows the convergence of $\Psi(t)$ to boundary measure $\mu_{13}(.)$ on S_2.

8 Discussions and Conclusions

A deterministic was model studied in reference [1] with six equilibrium points p_0, p_1, p_2, p_3, p_4 and p_5. In this model we have studied $\mu(.)$, $\mu_1(.)$, $\mu_{12}(.)$, $\mu_{13}(.)$ and $\nu(.)$ ergodic invariant measures. We consider that the equilibrium point and ergodic invariant measure related as follows:

(i) On invariant set $\{0,0,0\}$, p_0 is equivalent to measure $\mu(.)$.

(ii) On invariant set S^1, p_1 equivalent to measure $\mu_1(.)$.

(iii) Since the set S_3 has no meaning, therefore p_2 has no equivalent set.

(iv) On invariant set S_1, p_3 is equivalent to measure $\mu_{12}(.)$.

(v) On invariant set S_2, p_4 is equivalent to measure $\mu_{13}(.)$.

(vi) On invariant set S, p_5 is equivalent to measure $\nu(.)$.

We studied and compared stochastic and deterministic models and analyzed their stability. We also studied how environmental noise affect the survivability? From reference [1] find the equilibrium point p_0 is not stable in the model. But measure $\mu(.)$ is stable for $v_1(\mu) < 0$ in our stochastic model. If white noise β_1 is enough large, the equilibrium point p_0 may stable.

Thus, we get the idea that driving force which is produced by environment affect the system and it could system to extinction from partial extinction. From this we find the significance of drive force which is caused by environment. Asymptotically stable equilibrium points p_2 which is unconditional for the model from the literature [1], the equilibrium points p_2 is locally stable. Hence, it can not be globally stable. In our stochastic model the invariant subset S_3 is meaningless.

Under some certain conditions the equilibrium points p_1, p_3, p_4 and p_5 may globally asymptotically stable in model (1.1). In same manner the ergodic invariant measures $\mu_1(.), \mu_{12}(.), \mu_{13}(.)$ and $\nu(.)$ are for stochastic model (2) under some certain conditions. Hence we find that that stochastic model also possesses the properties of deterministic model. We studied and find that the survivability of the system depend on dynamic bifurcation points. The parameters which are we using in the system may affect the dynamic bifurcation points. Apart from this we also find that environmental noise, intrinsic growth and competition coefficients affect the system. The motivation of the paper is that we can apply some restrictions on scavenger or some special kinds of scavenger to use for the further study of the model.

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Conflicts of Interest The authors declare no conflict of interest.

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