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Estimating the Second Order Hankel Determinant for the Subclass of Bi-Close-to-Convex Function of Complex Order

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Abstract

The article aims to estimate the coefficient bounds for the second Hankel determinant by using the class of bi-close-to-convex functions of a complex order in the open unit disk. Making the direct application of Carathéodory function along with the closely related properties of starlike functions, we obtain the upper bound for the second Hankel determinant via certain subclass of bi-close-to-convex functions of complex order. The study discusses the maximization of the second Hankel determinant in both conventional graph and analytic methods. Moreover, we explore and modify some results on the study of bi-close-to-convex functions and its second degree Hankel determinant. At the end of the article, we remark on improvement in the earlier work of some researchers and discover a better value than the one they obtained.

Keywords: Hankel determinant; coefficient bounds; analytic functions; univalent functions; bi-univalent functions; close-to-convex functions; bi-close-to-convex functions; Carathéodory function.

1 Introduction

Let the class of analytic functions be denoted by $\mathcal{A}$ in the open unit disk such that, $\Delta = \{|z| < 1 : z \in \mathbb{C}\}$. The class of analytic functions is defined by the following mathematical form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

A domain function mapping from a unit disk onto a complex plane is called univalent on a unit disk if for each $z_1,\ z_2 \in \Delta$ the function $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$. Let T represents the collection of sub-classes of analytic functions that are made up of functions that yields a unique value in the unit disk of a radius less than 1. The most pertinent illustration of such a univalent function is the Köebe's function, $K(z) = z/(1-z)^2$. The Köebe's one-quarter theorem is widely known for ensuring that the image of a unit disk contains a circle with a radius of $1/4$ for any univalent function $f \in T$. As a result, any univalent function $f \in \mathcal{A}$ has a corresponding inverse function, f^{-1} in a way that $f^{-1}(f(z)) = z$, $(z \in \Delta)$ and $f(f^{-1}(w)) = w$, $(|w| < 1/4)$. The Köebe one-quarter theorem verifies that the inverse of these functions, $f^{-1}(w) = \mathcal{H}(w)$, may be easily demonstrated by the following equality,

$$\mathcal{H}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots. \tag{2}$$

The study on the class of close-to-convex functions was initiated by Kaplan[18] which states that, a function $f(z) \in \mathcal{A}$ is called close-to-convex function in the unit disk if there exists a star-like function g, consequently the following condition holds:

$$\mathfrak{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \qquad (z \in \Delta). \tag{3}$$

For the sake of equality equation (3), can be expressed as: $zf'(z) = p(z)g(z)$, with $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} h_n z^n. \tag{4}$$

Wherefore, a function $\mathcal{H}(w) \in \mathcal{A}$ is called the inverse of close-to-convex $(\mathfrak{C})$ functions in the unit disk assuming there is a function $G \in S^*$ such that $g^{-1}(w) = G(w)$ and $f^{-1}(w) = \mathcal{H}(w)$ then the following inequality holds:

$$\mathfrak{Re}\left\{\frac{w\mathcal{H}'(w)}{G(w)}\right\} > 0, \qquad (z \in \Delta), \tag{5}$$

equivalently (5) yields $w\mathcal{H}'(w) = q(w)G(w)$, with $G(w) = w + \sum_{n=2}^{\infty} h_n w^n$. It is known from [12, 39], that a function f is known to be close-to-convex with the argument ϕ if a real number $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ operates simultaneously with starlike functions g and G respectively in the following manner,

$$\mathfrak{Re}\left(\frac{e^{i\phi}zf'(z)}{g(z)}\right) > 0 \quad \text{and} \quad \mathfrak{Re}\left(\frac{e^{i\phi}w\mathcal{H}'(w)}{G(w)}\right) > 0, \qquad (z \in \Delta). \tag{6}$$

Al-Amiri and Thotage [1] introduced the class of close-to-convex functions of the complex order b, where b is a nonzero complex number. A function f with a complex order $b \neq 0$ is considered to be close-to-convex function , if the following inequality holds:

$$\mathfrak{Re}\left[1+\frac{1}{b}\left\{\left(\frac{zf'(z)}{g(z)}\right)-1\right\}\right]>0, \qquad (z\in\Delta), \tag{7}$$

for $g \in S^*$ (the starlike function).

A function f is known to be bi-univalent if the analytic function and its inverse are both univalent in the unit disc. For example $f(z) = -log(1-z)$ and $f(z) = z/1-z$. Analytically a function f belongs to the class of bi-close-to-convex function if the expression (3) and (5) hold simultaneously. The class of bi-close-to-convex function ($\mathfrak{C}_\Sigma$) of order α $(0 \le \alpha < 1)$ was studied by [15]. The class of bi-close-to-convex function fulfills the following conditions:

$$\mathfrak{Re}\left(\frac{zf'(z)}{g(z)}\right)>\alpha, \quad \text{and} \quad \mathfrak{Re}\left(\frac{w\mathcal{H}'(w)}{G(w)}\right)>\alpha, \qquad \left(z\in\Delta, \mathcal{H}=f^{-1}(z)\right). \tag{8}$$

The class Σ of bi-univalent functions was studied by Lewin in 1967 (see [21]). In this study he estimated the second coefficient bound ($|a_2|$) to be less than 1.51. Consequently other researchers such as [5] published that $|a_2| \le \sqrt{2}$ and some proved that the maximum value of $|a_2| = \frac{4}{3}$, (see [24]). Further, Tan [38] showed that $|a_2| < 1.485$, which is regarded as the most accurate estimate available for bi-univalent functions. In a progression in the study of bi-univalent function Srivastava et al. [35] have introduced two subclasses whose inverse has one-one analytic continuation towards the open unit disc. In 2017, Sakar and Güney [32] investigated the m-fold symmetric analytic functions and found coefficients for analytic bi-univalent functions defined by fractional calculus through the application of Faber polynomial expansion. The study on bi-univalent functions has been a hot topic for the past several years see for example [22, 37]. The coefficient problems of the form $\{|a_n| : n \ge 2\}$ for the functions $f \in \Sigma$ defined by (1) is apparently open problems and under research for sharp bounds. In recent study of bi-univalent functions, Rehman et al. [31] have made an attempt to solve the $n-$th coefficient bound for a specific class of bi-univalent functions. Srivastava et al. [34] applied the Faber polynomial expansion method to estimate the coefficient of general Taylor-Maclaurin series and the Fekete-Szegö type inequalities for the class of bi-close-to-convex function. A new subclass of bi-close-to-convex functions associated with the generalized hypergeometric functions, q-fractional derivative operator and with bounded boundary rotation is recently studied by [6, 36] and coefficient estimates of bi-close-to-convex functions associated with generalized hypergeometric functions for Faber polynomial were studied by Jie et al. [41].

The Hankel determinant works as a tool in the study of univalent functions, for instance in demonstrating how a function of bounded properties behaves in a unit disc Δ. This means that the function with Laurent series in the vicinity of the origin with an integral coefficient is rational by having a reasonable ratio between two bounded analytic functions. The q^{th} Hankel determinant for $q \ge 1$ and $n \ge 0$ was developed by Noonan and Thomas [25], and it states

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}. \tag{9}$$

The Hankel determinant plays a vital role in the study of singularity see for example Danies [8] and Edrei [10]. Other researchers who also considered this determinant for their studies such as Noor [27] calculated the increasing rate of $H_q(n)$ for the functions f specified by (1) whenever

n approaches the infinity with a constrained boundary rotation. Pommerenke [29] investigated the Hankel determinants for starlike functions, univalent functions, and areally mean p-valent functions. In another piece of research Pommerenke [30] shown that the Hankel determinants of univalent functions meet the following inequality,

$$H_q(n) < Kn^{-(\lambda+\frac{1}{2})q+\frac{3}{2}}, \qquad (n = 1, 2, \cdots; \quad q = 2, 3, \cdots; \quad \lambda > 1/4000),$$

where K relies on q.

Moreover, [16] proved that for the areally mean univalent functions $H_2(n) < An^{\frac{1}{2}}$ $(n = 1, 2, \cdots)$ and A is an absolute constant. There are researchers who further studied the Hankel determinant of univalent functions and, Coefficient differences and Hankel determinants of areally mean p-valent functions see [11, 26]. Note that for $q = 2$ and $n = 1$, the above Hankel determinant interacts with the well-known result of Fekete-Szegö functional $|a_3 - a_2^2| = H_2(1)$. This function was extended by introducing μ as real and complex both. The sharp estimates for μ real were estimated by Fekete-Szegö,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 4\mu - 3, & \mu \leq 1, \\ 1 + 2exp\left(\dfrac{-2}{1-\mu}\right), & 0 \leq \mu \leq 1, \\ 3 - 4\mu, & \mu \geq 0. \end{cases} \tag{10}$$

The analogues investigation of (10), studied by Keogh and Merkes [19], states that for ϕ and λ be real numbers, where $|\phi| < \dfrac{\pi}{2}$, $\lambda \in [0, 1)$ and let $S^*(\phi, \lambda)$ is the set of analytic functions in the open unit disk such that $f(0) = 0$ and $f'(0) = 1$ then $f \in S^*(\phi, \lambda)$ if the following condition holds:

$$\mathfrak{Re}\left(e^{i\phi}\frac{zf'(z)}{f(z)}\right) > \lambda\cos\phi, \qquad (z \in \Delta). \tag{11}$$

In addition, Zaprawa [40] investigated some more classes of bi-univalent functions related to Fekete-Szegö problem. Considering the inverse function, $(f^{-1}(w) = w + \sum_{k=2}^{\infty}\gamma_k w^k)$ to the strongly starlike functions of order $\alpha(\alpha \in (0, 1])$, Ali [2] estimated the first four sharp coefficients bounds for the Fekete-Szegö functional $(|\gamma_3 - t\gamma_2^2|)$ where t is a real number. In the recent studies regarding the second degree Hankel determinant $(|a_2a_4 - a_3^2|)$, Lee et al.[20] obtained the functions associated to subclasses of Ma-Minda convex and starlike functions. Further Janteng [17] was able to calculate the sharp estimates for the second-degree Hankel determinant by adopting the function whose derivative contains a positive real part.

2 Preliminary Results and Discussion

In this section we include the definitions and lemmas which are necessary to establish our main results. The portion includes the construction of a new sub-class of bi-close-to-convex functions and its relation to other classes studied by other researchers.

Definition 2.1. *The class of Carathéodory functions P, is the class of functions $p \in \mathcal{A}$ of the form,*

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n, \qquad (z \in \Delta).$$

In view of Carathéodory functions P, the conducive results for the functions of the class P are:

Lemma 2.1. [28] *If the function $p \in P$ is given in the form (1), then the sharp estimates $|b_n| \leq 2$, $(n = 1, 2, \cdots)$ holds.*

Lemma 2.2. [13] *If the function $p \in P$ is given in the form (1), then*

$$2b_2 = b_1^2 + (4 - b_1^2)x, \tag{12}$$

and

$$4b_3 = b_1^3 + 2(4 - b_1^2)b_1x - b_1(4 - b_1^2)x^2 + 2(4 - b_1^2)\left(1 - |x|^2\right)t, \tag{13}$$

for some x, t with $(|x|, |t| \leq 1)$.

Lemma 2.3. [19] *Let the function $f \in S^*$ be defined by (4). Then for any real number μ,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if} \quad \mu \leq \frac{1}{2}, \\ 1, & \text{if} \quad \frac{1}{2} \leq \mu \leq 1, \\ 4\mu - 3, & \text{if} \quad \mu \geq 1. \end{cases}$$

Lemma 2.4. [17] *Let the function $f \in S^*$ be defined by (4). Then $|a_2a_4 - a_3^2| \leq 1$.*

Lemma 2.5. [4] *Let the function $f \in S^*$ be defined by (4). Then $|a_2a_3 - a_4| \leq 2$.*

Remark 2.1. *Note that equality in Lemma 2.4 and Lemma 2.5, is obtained for the Köebe function, $K(z) = z/(1 - z)^2$.*

Lemma 2.6. [9] *Let the function $f \in S^*$ be defined by (3), then $|a_n| \leq n\,(n = 2, 3, \cdots)$. The inequality is strictly true for all n, unless f is the rotation of Köebe function, $K(z) = z/(1 - z)^2$.*

We now introduce the sub-class of bi-close-to-convex function of a complex order m, for which we seek the upper bound for the second order Hankel determinant for functions belonging to the class $\mathfrak{C}_\Sigma(m, \phi, \lambda)$. The functional for the second order Hankel determinant is given by:

$$|a_2a_4 - a_3^2| = H_2(2).$$

Hence, in view of (3), (5)–(7) and (11), we construct a subclass $\mathfrak{C}_\Sigma(m, \phi, \lambda)$, that is defined in the following manner.

Definition 2.2. *A function $f \in \mathcal{A}$ is said to be in the class $\mathfrak{C}_\Sigma(m, \phi, \lambda)$ if it satisfies the following conditions:*

$$\mathfrak{Re}\left[1 + \frac{1}{m}\left\{\frac{e^{i\phi}zf'(z)}{g(z)} - 1\right\}\right] > \lambda\cos\phi, \qquad (z \in \Delta), \tag{14}$$

and

$$\mathfrak{Re}\left[1 + \frac{1}{m}\left\{\frac{e^{i\phi}w\mathcal{H}'(w)}{G(w)} - 1\right\}\right] > \lambda\cos\phi, \qquad (w \in \Delta), \tag{15}$$

where $\mathcal{H}(w) = f^{-1}(w)$ and $G(w) = g^{-1}(w)$ defined by (2), $g(z) = z + \sum_{n=2}^{\infty} h_nz^n \in S^*$ and $G(w) = w + \sum_{n=2}^{\infty} h_nw^n \in S^*$ with $(m \in \mathbb{C}, m \neq 0)$, $\phi \in (-\pi/2, \pi/2)$ and $(0 \leq \lambda < 1)$.

It is easy to see that by specifying values to m, ϕ and λ, this subclass $\mathfrak{C}_\Sigma(m, \phi, \lambda)$, interacts with some substantial subclasses that have been studied by numerous authors in their earlier works, for instance we enlist the following relations;

(i) $\mathfrak{C}(m,0,0) \equiv K(b)$ (Al-Amiri and Thotage [1]).

(ii) $\mathfrak{C}(1,0,\lambda) \equiv S(\alpha)$ (Hamidi and Jahangiri [15]).

(iii) $\mathfrak{C}(1,0,0) \equiv C$ (Kaplan et al. [18]).

(iv) $\mathfrak{C}_\Sigma(1,0,0) \equiv C_\Sigma$ (Güney et al. [14]).

(v) $\mathfrak{C}(m,0,\lambda) \equiv S^*(\gamma,\beta)$ (Altıntaş et al. [3]).

(vi) $\mathfrak{C}(m,0,0) \equiv S^*(b)$ (Nasr and Aouf [23]).

(vii) $\mathfrak{C}(1,\gamma,0) \equiv C_\beta(g)$ (Wieclaw and Zaprawa [39]).

(viii) $\mathfrak{C}(1,\gamma,\lambda) \equiv S(\gamma,\lambda)$ (Keogh and Merkes [19]).

Recently, Cho et al. [7] and Güney et al. [14], investigated the class of bi-close-to-convex function and respectively obtained the maximum bounds $\frac{227}{36}$ and $\frac{353}{36}$, for the second Hankel determinant $H_2(2)$. In a recent studies [33] has considered a subclass of bi-close-to-convex functions to obtain the initial estimates for the functions in these subclasses.

The main goal of this investigation is to obtain a smaller upper bound for the second Hankel determinant via our newly defined subclass of bi-close-to-convex function of a complex order.

3 Main Results

Theorem 3.1. *Let the function $f(z)$ given by (1) be in the class $\mathfrak{C}_\Sigma(m,\phi,\lambda)$ and $G(w) = g^{-1}(w)$. Then,*

$$|H_2(2)| = |a_2a_4 - a_3^2| \leq \frac{145}{18}.$$

Proof. Let $f \in \mathfrak{C}_\Sigma(m,\phi,\lambda)$ and $\mathcal{H}(w) = f^{-1}(w)$. Then by using (14) and (15), we receive

$$e^{i\phi}\left(\frac{zf'(z)}{g(z)}\right) = \{1 + m(1-\lambda\cos\phi)(p(z)-1)\}, \qquad (\forall\, z \in \Delta), \tag{16}$$

and

$$e^{i\phi}\left(\frac{w\mathcal{H}'(w)}{G(w)}\right) = \{1 + m(1-\lambda\cos\phi)(q(w)-1)\}, \qquad (\forall\, w \in \Delta), \tag{17}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \in P, (z \in \Delta)$ and $q(w) = 1 + \sum_{n=1}^{\infty} c_n w^n \in P, (w \in \Delta)$.

Then, upon comparing the coefficients of (16) and (17) we have,

$$2a_2 - h_2 = \psi b_1, \tag{18}$$

$$3a_3 - 2a_2h_2 + h_2^2 - h_3 = \psi b_2, \tag{19}$$

$$4a_4 - 3a_3h_2 - 2a_2h_3 + 2a_2h_2^2 + 2h_2h_3 - h_2^3 - h_4 = \psi b_3, \tag{20}$$

and

$$-2a_2 + h_2 = \psi c_1, \tag{21}$$
$$6a_2^2 - 3a_3 + h_3 - 2a_2h_2 - h_2^2 = \psi c_2, \tag{22}$$
$$-4a_4 - 3a_3h_2 - 2a_2h_3 + 2a_2h_2^2 - 3h_2h_3 + 2h_2^3 + h_4 + 20a_2a_3 + 6a_2^2h_2 - 20a_2^3 = \psi c_3, \tag{23}$$

where $\psi = e^{-i\phi}m(1-\lambda\cos\phi)$, is used to shorten the lengthy expressions. Note that by simplifying the equations (18) and (21) we obtain $b_1 = -c_1$. Thus making the use of equation (18) or (21), we get

$$a_2 = \frac{h_2 + b_1\psi}{2}. \tag{24}$$

Next on subtracting (22) from ((19) with the use of (24), we receive a_3;

$$a_3 = \frac{h_3}{3} - \frac{h_2^2}{12} + \frac{h_2b_1\psi}{2} + \frac{b_1^2\psi^2}{4} + \frac{(b_2-c_2)\psi}{6}. \tag{25}$$

Further upon subtracting (23) from (20), and then utilizing the equations (24) and (25), we get a_4

$$\begin{aligned} a_4 = {} & \frac{h_4}{4} + \frac{7h_2^3}{48} - \frac{5h_2h_3}{24} - \frac{h_2^2b_1\psi}{24} + \frac{3h_2b_1^2\psi^2}{16} + \frac{5b_3b_1\psi}{12} + \frac{5h_2(b_2-c_2)\psi}{24} \\ & + \frac{5b_1(b_2-c_2)\psi^2}{24} + \frac{(b_3-c_3)\psi}{8}. \end{aligned} \tag{26}$$

At this stage, we can proceed to compute the second Hankel determinant $H_2(2)$, just by making some necessary simplification and setting terms in a way so that Lemmas (2.1 and 2.2) are functional to them. Then by applying the modulus function we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| = {} & \bigg| \frac{19}{144}h_2^2(b_2-c_2)\psi - \frac{1}{9}h_3(b_2-c_2)\psi - \frac{1}{36}(b_2-c_2)^2\psi^2 + \frac{1}{16}h_2(b_3-c_3)\psi \\ & + \left(\frac{1}{8}h_2h_4 - \frac{1}{9}h_3^2\right) + \left(\frac{1}{8}h_4 - \frac{11}{48}h_2h_3\right)b_1\psi + \left(\frac{1}{24}h_3 - \frac{13}{96}h_2^2\right)b_1^2\psi^2 \\ & + \left(\frac{19}{288}h_2^4 - \frac{7}{144}h_2^2h_3\right) + \left(\frac{13}{96}h_2^3b_1\psi + \frac{1}{24}h_2b_1(b_2-c_2)\psi^2 - \frac{5}{32}h_2b_1^3\psi^3\right) \\ & + \left(\frac{1}{16}b_1(b_3-c_3)\psi^2 + \frac{1}{48}b_1^2(b_2-c_2)\psi^3 - \frac{1}{16}b_1^4\psi^4\right)\bigg|. \end{aligned} \tag{27}$$

Setting equation (27) for the inequality properties defined by Lemmas 2.3 – 2.6, we have

$$\begin{aligned} |a_2a_4 - a_3^2| = {} & \bigg| \frac{1}{8}\left(h_2h_4 - h_3^2\right) + \frac{1}{72}h_3^2 - \frac{1}{9}\left(h_3 - \frac{19}{16}h_2^2\right)(b_2-c_2)\psi - \frac{1}{36}(b_2-c_2)^2\psi^2 \\ & + \frac{1}{16}h_2(b_3-c_3)\psi + \frac{1}{8}(h_4 - h_2h_3)b_1\psi - \frac{5}{48}h_2h_3b_1\psi \\ & + \frac{1}{24}\left(h_3 - \frac{13}{4}h_2^2\right)b_1^2\psi^2 - \frac{7}{144}\left(h_3 - \frac{19}{14}h_2^2\right)h_2^2 \\ & - \frac{5}{32}h_2b_1\psi\left\{b_1^2\psi^2 - \frac{4}{15}(b_2-c_2)\psi - \frac{13}{15}h_2^2\right\} \\ & - \frac{1}{16}b_1\psi^2\left\{b_1^3\psi^2 - \frac{1}{3}b_1(b_2-c_2)\psi - (b_3-c_3)\right\}\bigg|. \end{aligned} \tag{28}$$

Remark 3.1. *For $\psi = 1$, equation (28) matches with equation (2.20) obtained by Güney et al. [14, page 8] and for $\psi = 2$, coincides with equation (15) of Cho et al. [7, page 5].*

In order to determine the values of $(b_2 - c_2)$ and $(b_3 - c_3)$, especially when the function is defined by starlike function, then by virtue of Lemma 2.2 with $b_1 = -c_1$ we get,

$$\left.\begin{aligned} 2b_2 &= b_1^2 + (4 - b_1^2)x \\ 2c_2 &= c_1^2 + (4 - c_1^2)y \end{aligned}\right\} \Longrightarrow b_2 - c_2 = \left(\frac{4 - b_1^2}{2}\right)(x - y), \tag{29}$$

and for some minimum values of $\{|t|, |v|, |x|, |y|\} \leq 1$, we can consider

$$\left.\begin{aligned} 4b_3 &= b_1^3 + 2(4 - b_1^2)b_1 x - b_1(4 - b_1^2)x^2 + 2(4 - b_1^2)\left(1 - |x|^2\right)t \\ 4c_3 &= c_1^3 + 2(4 - c_1^2)c_1 y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)\left(1 - |y|^2\right)v \end{aligned}\right\}. \tag{30}$$

Then subtracting the above pair of equations we receive $(b_3 - c_3)$,

$$\begin{aligned} b_3 - c_3 = \frac{b_1^3}{2} &+ \frac{b_1(4 - b_1^2)}{2}(x + y) - \frac{b_1(4 - b_1^2)}{4}(x^2 + y^2) \\ &+ \frac{(4 - b_1^2)}{2}\left\{(1 - |x|^2)t - (1 - |y|^2)v\right\}. \end{aligned} \tag{31}$$

In view of equations from (29)-(31), the equation (28) becomes

$$\begin{aligned} |a_2 a_4 - a_3^2| = \Bigg| &\frac{1}{8}\left(h_2 h_4 - h_3^2\right) + \frac{1}{72}h_3^2 + \frac{1}{8}\psi b_1\left(h_4 - h_2 h_3\right) - \frac{5}{48}h_2\left(h_3 - \frac{13}{10}h_2^2\right)\psi b_1 \\ &+ \frac{1}{24}\left(h_3 - \frac{13}{4}h_2^2\right)\psi^2 b_1^2 - \frac{7}{144}h_2^2\left(h_3 - \frac{19}{14}h_2^2\right) - \frac{5}{32}h_2\psi^3 b_1^3 \\ &+ \frac{1}{32}\psi^2 b_1^4 + \frac{1}{32}h_2\psi b_1^3 - \frac{1}{16}\psi^4 b_1^4 + (x + y)\left\{\frac{1}{32}\psi^2 b_1^2(4 - b_1^2) + \frac{1}{32}h_2\psi b_1(4 - b_1^2)\right\} \\ &+ (x - y)\left\{\frac{1}{96}\psi^3 b_1^2(4 - b_1^2) + \frac{1}{48}h_2\psi^2 b_1(4 - b_1^2) - \frac{1}{18}\psi(4 - b_1^2)\left(h_3 - \frac{19}{16}h_2^2\right)\right\} \\ &- \frac{1}{144}\psi^2(4 - b_1^2)^2(x - y)^2 + (x^2 + y^2)\left\{-\frac{1}{64}\psi^2 b_1(4 - b_1^2) - \frac{1}{64}h_2\psi b_1(4 - b_1^2)\right\} \\ &+ (1 - |x|^2)t\left\{\frac{1}{32}\psi^2 b_1(4 - b_1^2) + \frac{1}{32}h_2\psi(4 - b_1^2)\right\} \\ &+ (1 - |y|^2)v\left\{-\frac{1}{32}\psi^2 b_1(4 - b_1^2) - \frac{1}{32}h_2\psi(4 - b_1^2)\right\}\Bigg|. \end{aligned} \tag{32}$$

Once again quoting the Lemma 2.1, that $|b_1| \leq 2$ so, b_1 can be replaced by b, while assuming that

$b \in [0, 2]$, then by triangle inequality the right hand side of equation (32) becomes,

$$\begin{aligned}|a_2a_4 - a_3^2| \le\ & \frac{1}{8}|h_2h_4 - h_3^2| + \frac{1}{72}|h_3|^2 + \frac{1}{8}|\psi||h_4 - h_2h_3|b + \frac{5}{48}|h_2||h_3 - \frac{13}{10}h_2^2||\psi|b \\ & + \frac{1}{24}|h_3 - \frac{13}{4}h_2^2||\psi|^2b^2 + \frac{7}{144}|h_2|^2|h_3 - \frac{19}{14}h_2^2| + \frac{5}{32}|h_2||\psi|^3b^3 + \frac{1}{32}|\psi|^2b^4 \\ & + \frac{1}{32}|h_2||\psi|b^3 + \frac{1}{16}|\psi|^4b^4 + \frac{1}{144}|\psi|^2(4-b^2)^2(|x|+|y|)^2 \\ & + (|x|+|y|)\left\{\frac{1}{32}|\psi|^2b^2(4-b^2) + \frac{1}{32}|h_2||\psi|b(4-b^2) + \frac{1}{96}|\psi|^3b^2(4-b^2)\right. \\ & \left. + \frac{1}{48}|h_2||\psi|^2b(4-b^2) + \frac{1}{18}|\psi|(4-b^2)|h_3 - \frac{19}{16}h_2^2|\right\} \\ & + (|x|^2+|y|^2)\left\{\frac{1}{64}|\psi|^2b(4-b^2) + \frac{1}{64}|h_2||\psi|b(4-b^2)\right\} \\ & + (1-|x|^2)t\left\{\frac{1}{32}|\psi|^2b(4-b^2) + \frac{1}{32}|h_2||\psi|(4-b^2)\right\} \\ & + (1-|y|^2)v\left\{\frac{1}{32}|\psi|^2b(4-b^2) + \frac{1}{32}|h_2||\psi|(4-b^2)\right\}.\end{aligned} \tag{33}$$

Now manipulating the Lemmas from 2.3-2.6, so that we can obtain

$$\begin{aligned}|a_2a_4 - a_3^2| \le\ & \frac{13}{18} + \frac{7}{12}|\psi|b + \frac{5}{12}|\psi|^2b^2 + \frac{5}{16}|\psi|^3b^3 + \frac{1}{16}|\psi|^4b^4 + \frac{1}{32}|\psi|(2+|\psi|b)(b^3-2b^2+8) \\ & + (|x|+|y|)\left\{\frac{1}{96}|\psi|^2(3+|\psi|)b^2(4-b^2) + \frac{1}{48}|\psi|(3+2|\psi|)b(4-b^2) + \frac{7}{72}|\psi|(4-b^2)\right\} \\ & + (|x|^2+|y|^2)\left\{\frac{1}{64}|\psi|\{4-(2-|\psi|)b\}(b^2-4)\right\} + \frac{1}{144}|\psi|^2(4-b^2)^2(|x|+|y|)^2.\end{aligned} \tag{34}$$

Thus setting $\alpha = |x| \le 1$, $\gamma = |y| \le 1$ and $|a_2a_4 - a_3^2| = H_2(2)$, we get

$$H_2(2) \le Q_1 + (\alpha+\gamma)Q_2 + (\alpha^2+\gamma^2)Q_3 + (\alpha+\gamma)^2Q_4 = F(\alpha,\gamma) \tag{35}$$

where, $\{Q_1, Q_2, Q_3, Q_4\}$ respectively represents;

$$\begin{aligned}Q_1(b,\psi) &= \left[\frac{13}{18} + \frac{7}{12}|\psi|b + \frac{5}{12}|\psi|^2b^2 + \frac{5}{16}|\psi|^3b^3 + \frac{1}{16}|\psi|^4b^4 + \frac{1}{32}|\psi|(2+b|\psi|)(b^3-2b^2+8)\right] \ge 0, \\ Q_2(b,\psi) &= \left[\frac{1}{96}|\psi|^2(3+|\psi|)b^2(4-b^2) + \frac{1}{48}|\psi|(3+2|\psi|)b(4-b^2) + \frac{7}{72}|\psi|(4-b^2)\right] \ge 0, \\ Q_3(b,\psi) &= \left[-\frac{1}{64}|\psi|\{4-(2-|\psi|)b\}(4-b^2)\right] \le 0, \\ Q_4(b,\psi) &= \left[\frac{1}{144}|\psi|^2(4-b^2)^2\right] \ge 0.\end{aligned} \tag{36}$$

We now investigate the maximization of the function $F(\alpha, \gamma)$ over the closed square region of the $\mathbb{B} = \{(\alpha, \gamma) : 0 \le (\alpha, \gamma) \le 1\}$ for $b \in [0, 2]$ and $|\phi| < \pi/2$. For this reason we have to discuss the maximization of $F(\alpha, \gamma)$ for several cases in focus when $b = 0$, $b = 2$, $b \in (0, 2)$ and $\phi = 0$, $\phi = \pi/2$ and $\phi \in (-\pi/2, \pi/2)$. It is evident in Figure 1 that for $b = 2$ and $\phi = 0$, the maximum value is just ahead of 8.

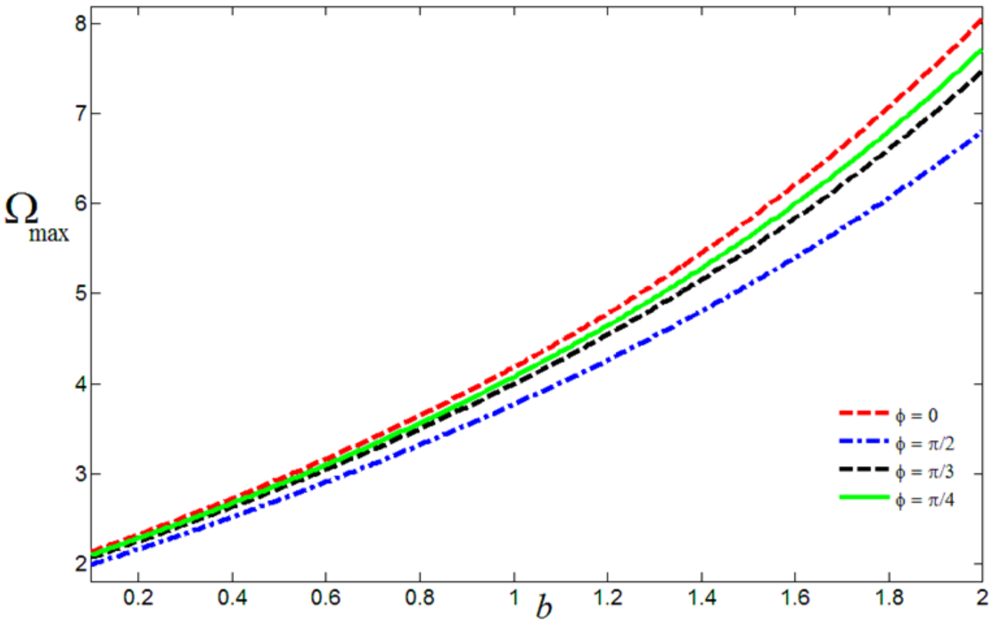

Figure 1: Maximization of $F(\alpha,\gamma) = \Omega_{\max}$.

Analytically by looking at the expressions in (36), we see that $Q_3 < 0$ and $Q_3 + 2Q_4 > 0$ for $b \in (0, 2)$, consequently we get

$$F(\alpha,\gamma) = F_{\alpha\alpha}.F_{\gamma\gamma} - (F_{\alpha\gamma})^2 < 0.$$

Thereupon, the local maximum of $F(\alpha,\gamma)$ cannot occur inside the square region $\mathbb{B}$. In order to examine the upper bound of the function $F(\alpha,\gamma)$ on the edge of square $\mathbb{B}$, we assume $\alpha = 0$ and $0 \le \gamma \le 1$ (or $\gamma = 0$ and $0 \le \alpha \le 1$). Thus with regards to $F(\alpha,\gamma) = F(\gamma,\alpha)$, it is fair enough to verify that there exists a maximum of

$$F(\alpha,\gamma) = F(\alpha,\alpha) = Q_1 + 2\alpha Q_2 + 2\alpha^2(Q_3 + 2Q_4), \tag{37}$$

on $\alpha \in [0, 1]$, according to $b \in [0, 2]$. Then on differentiating $F(\alpha)$, we get

$$F'(\alpha) = 2Q_2 + 4\alpha(Q_3 + 2Q_4). \tag{38}$$

Here we need to show that $F(\alpha)$ is an increasing function. It is easy to see that if $Q_3 + 2Q_4 \ge 0$, then $F'(\alpha) > 0$ for $\alpha > 0$, consequently $F(\alpha)$ is an increasing function and an increasing function cannot have a local maxima in the interior of a closed region of $\mathbb{B}$. For $\alpha \in (0, 1)$ and for any fixed b, such that $b \in [0, 2]$ the maximum value of the function $F(\alpha)$ appears at $F(1)$,

$$\max\{F(\alpha)\} = F(1) = Q_1 + 2Q_2 + 2Q_3 + 4Q_4. \tag{39}$$

Secondly for the case if $Q_3 + 2Q_4 < 0$, we follow the work of [7] and accordingly consider the equation (37) for $0 < \alpha < 1$ with $b \in [0, 2]$, we consider for a critical point by setting $F'(\alpha) = 0$ in (38),

$$\begin{aligned}\alpha_\circ &= \frac{-Q_2}{2(Q_3 + 2Q_4)} = \frac{Q_2}{2k} \\ &= \frac{2\left(3b^2|\psi|^2 + 3b(3b+4)|\psi| + 18b + 28\right)}{(8b^2 - 9b - 32)|\psi| + 36} > 1,\end{aligned} \tag{40}$$

for any fixed $b \in [0,2]$, where $k = -(Q_3 + 2Q_4) > 0$. This means for $\alpha_\circ = \dfrac{Q_2}{2k} > 1$, It follows that $k < \dfrac{Q_2}{2} \leq Q_2$, and finally we can state that $Q_2 + Q_3 + 2Q_4 \geq 0$, and hence proved an increasing function. Therefore, we can write

$$F(0) = Q_1 \leq Q_1 + 2(Q_2 + Q_3 + 2Q_4) = F(1).$$

Since, $F_{\max}(\alpha) = F(1)$, coincides with (39), this implies $F_{\max}(\alpha,\gamma) = F(1,1)$ lies on the boundary of the square. Let $\Omega : (0,2) \longrightarrow \mathbb{R}$ be defined by,

$$\Omega(b) = F(1,1) = Q_1 + 2Q_2 + 2Q_3 + 4Q_4.$$

Thus, using the values of $\{Q_1, Q_2, Q_3, Q_4\}$ defined by (36), we receive

$$\begin{aligned}\Omega(\psi,b) = &\frac{1}{288}|\psi|\left(18|\psi|^3 - 6|\psi|^2 - 25|\psi| - 36\right)b^4 + \frac{1}{32}|\psi|^2\left(10|\psi| - 1|\right)b^3 \\ &+ \frac{1}{36}|\psi|\left(3|\psi|^2 + 28|\psi| + 11\right)b^2 + \frac{1}{24}|\psi|\left(3|\psi| + 20\right)b + \frac{1}{18}\left(8|\psi|^2 + 14|\psi| + 13\right).\end{aligned} \tag{41}$$

By reverting $\psi = e^{-i\phi}m(1-\lambda\cos\phi)$ in (41) the function $\Omega(\psi,b)$ becomes,

$$\begin{aligned}\Omega(m,\phi,\lambda)(b) = &\frac{e^{4im(\phi)}|m(1-\lambda cos\phi)|^4}{16}b^4 - \frac{e^{3im(\phi)}|m(1-\lambda cos\phi)|^3}{48}b^2(b^2 - 15b - 4) \\ &- \frac{e^{2im(\phi)}|m(1-\lambda cos\phi)|^2}{288}(25b^4 + 9b^3 - 224b^2 - 36b - 128) \\ &- \frac{e^{im(\phi)}|m(1-\lambda cos\phi)|}{72}(9b^4 - 22b^2 - 60b - 56) + \frac{13}{18},\end{aligned} \tag{42}$$

where m is a non-zero complex number and $im(z)$ is the iota of a complex number and $|z|$ is the modulus function. For instance, let us consider the bounded value of m, in the open unit disk $m = 1 - \lambda$, $\lambda = 0$ and $\phi = 0$ in (42), we obtain an expression that yields algebraically and graphically same value (see Figure 1).

$$\Omega_{\max}(b) = \frac{35}{18} + \frac{23}{24}b + \frac{7}{6}b^2 + \frac{9}{32}b^3 - \frac{49}{288}b^4. \tag{43}$$

It is worth mentioning to see that $\Omega'(b) > 0$, with respect to b showing that $\Omega(b)$ is an increasing function of b. Therefore $\Omega(b)$ will attain the maximum value whenever $b = 2$. Thus the smallest maximum bound for the function defined by (43) is $\dfrac{145}{18} \approx 8.0\bar{5}$. This completes the asserted proof. □

Remark 3.2. *During the literature review of bi-close-to-convex functions, it was noted in the paper of Güney et al. [14, page 8] that the equality after the equation (2.20) in their work is further rectifiable. The original text in their work is given by:*

$$|a_2a_4 - a_3^2| = \left|\cdots + \frac{1}{8}\left(b_4 - b_2b_3\right)c_1 - \frac{5}{48}b_2b_3c_1 + \cdots + \frac{13}{96}b_2^3c_1 - \cdots\right|.$$

The resultant of above three coefficients of c_1 in [14, page 9] produces $\dfrac{47}{24}c$. Their result can be improved if these terms are combined for the applicability of Lemma 2.3. So, after combining the above like terms and then factor the common out from last two terms mentioned above, we receive

$$|a_2a_4 - a_3^2| = \left|\cdots + \frac{1}{8}\left(b_4 - b_2b_3\right)c_1 - \frac{5}{48}b_2\left(b_3 - \frac{13}{10}b_2^2\right)c_1 + \cdots\right|.$$

In view of Lemma 2.3, the new result obtained from the above expression is $\frac{17}{24}c$*. Following the replacement in their work till equation (2.22) the term becomes* $\frac{43}{24}c$*, the final estimation is* $\frac{223}{36}$*. Their result is* $\frac{353}{36}$*. The new estimation is far smaller than their work in [14] and the one determined by [7].*

4 Conclusion

In this article, we introduced a sub-class of bi-close-to-convex functions of a complex order. We determined the upper bound for the second Hankel determinant by considering the subclass of bi-close-to-convex function of a complex order. We enhanced the estimation of a maximum upper bound studied by [14] which is $\frac{353}{36}$ but after the modification the new upper bound for their work is $\frac{223}{36}$, which is more accurate then the value obtained in [14] and [7] which is $\frac{227}{36}$.

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References

[1] H. S. Al-Amiri & T. S. Fernando (1990). On close-to-convex functions of complex order. *International Journal of Mathematics and Mathematical Sciences*, *13*, Article ID: 840876. https://doi.org/10.1155/S0161171290000473.

[2] R. M. Ali, S. K. Lee, V. Ravichandran & S. Supramaniam (2012). Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. *Applied Mathematics Letters*, *25*(3), 344–351. https://doi.org/10.1016/j.aml.2011.09.012.

[3] O. Altıntaş, H. Irmak, S. Owa & H. M. Srivastava (2007). Coefficient bounds for some families of starlike and convex functions of complex order. *Applied Mathematics Letters*, *20*(12), 1218–1222. https://doi.org/10.1016/j.aml.2007.01.003.

[4] K. O. Babalola (2010). *Inequality Theory and Applications*, volume 6, chapter On $H_3(1)$ Hankel determinant for some classes of univalent functions, pp. 1–7. Nova Science Publishers, New York.

[5] D. A. Brannan & J. G. Clunie. Aspects of contemporary complex analysis. Proceedings of an Instructional Conference organized by the London Mathematical Society at the University of Durham, 1979 (a NATO Advanced Study Institute). London etc.: Academic Press, a Subsidiary of Harcourt Brace Jovanovich, Publishers 1980. https://zbmath.org/0483.00007.

[6] D. Breaz, P. Sharma, S. Sivasubramanian & S. M. El-Deeb (2023). On a new class of bi-close-to-convex functions with bounded boundary rotation. *Mathematics*, *11*(20), Article ID: 4376. https://doi.org/10.3390/math11204376.

[7] N. E. Cho, E. Analouei Adegani, S. Bulut & A. Motamednezhad (2019). The second Hankel determinant problem for a class of bi-close-to-convex functions. *Mathematics*, *7*(10), Article ID: 986. https://doi.org/10.3390/math7100986.

[8] P. Dienes (1957). *The Taylor Series: An Introduction to the Theory of Functions of a Complex Variable* volume 10. Dover Publications Inc, New York.

[9] P. L. Duren (2001). *Univalent Functions* volume 259 of *Grundlehren der mathematischen Wissenschaften*. Springer Science & Business Media, New York 1 edition. https://zbmath.org/0514.30001.

[10] A. Edrei (1939). On the recurrent terminators and the singularities of a function given by its Taylor development (translated through google translator). *Compositio Mathematica*, *7*, 20–88. https://zbmath.org/65.0308.02.

[11] M. M. Elhosh (1986). On the second Hankel determinant of univalent functions. *Bulletin of the Malaysian Mathematical Sciences Society*, *9*(1), 23–25. https://zbmath.org/0612.30017.

[12] A. W. Goodman & E. B. Saff (1978). On the definition of a close-to-convex function. *International Journal of Mathematics and Mathematical Sciences*, *1*, Article ID: 982173. https://doi.org/10.1155/S0161171278000150.

[13] U. Grenander & G. Szegö (1958). *Toeplitz Forms and Their Applications*. University of California Press, California. https://doi.org/10.1525/9780520355408.

[14] H. Ö. Güney, G. Murugusundaramoorthy & H. M. Srivastava (2019). The second Hankel determinant for a certain class of bi-close-to-convex functions. *Results in Mathematics*, *74*(3), Article ID: 93. https://doi.org/10.1007/s00025-019-1020-0.

[15] S. G. Hamidi & J. M. Jahangiri (2014). Faber polynomial coefficient estimates for analytic bi-close-to-convex functions. *Comptes Rendus Mathematique*, *352*(1), 17–20. https://doi.org/10.1016/j.crma.2013.11.005.

[16] W. K. Hayman (1968). On the second Hankel determinant of mean univalent functions. *Proceedings of the London Mathematical Society*, *3*(1), 77–94. https://doi.org/10.1112/plms/s3-18.1.77.

[17] A. Janteng, S. A. Halim & M. Darus (2006). Coefficient inequality for a function whose derivative has a positive real part. *Journal of Inequalities in Pure and Applied Mathematics*, *7*(2), 1–5.

[18] W. Kaplan (1952). Close-to-convex schlicht functions. *Michigan Mathematical Journal*, *1*(2), 169–185. https://doi.org/10.1307/mmj/1028988895.

[19] F. R. Keogh & E. P. Merkes (1969). A coefficient inequality for certain classes of analytic functions. *Proceedings of the American Mathematical Society*, *20*(1), 8–12. https://doi.org/10.1090/S0002-9939-1969-0232926-9.

[20] S. K. Lee, V. Ravichandran & S. Supramaniam (2013). Bounds for the second Hankel determinant of certain univalent functions. *Journal of Inequalities and Applications*, *2013*(1), 281. https://doi.org/10.1186/1029-242X-2013-281.

[21] M. Lewin (1967). On a coefficient problem for bi-univalent functions. In *Proceedings of the American Mathematical Society*, volume 18 pp. 63–68. JSTOR,. https://doi.org/10.2307/2035225.

[22] N. Mustafa, G. Mrugusundaramoorthy & T. Janani (2018). Second Hankel determinant for a certain subclass of bi-univalent functions. *Mediterranean Journal of Mathematics*, *15*, Article ID: 119. https://doi.org/10.1007/s00009-018-1165-1.

[23] M. A. Nasr & M. K. Aouf (1985). Starlike function of complex order. *Journal of Natural Sciences and Mathematics*, *25*(1), 1–12. https://zbmath.org/0596.30017.

[24] E. Netanyahu (1969). The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Archive for Rational Mechanics and Analysis*, *32*(2), 100–112. https://doi.org/10.1007/BF00247676.

[25] J. W. Noonan & D. K. Thomas (1976). On the second Hankel determinant of areally mean p-valent functions. *Transactions of the American Mathematical Society*, *223*, 337–346. https://doi.org/10.2307/1997533.

[26] J. W. Noonan (1974). Coefficient differences and Hankel determinants of areally mean p-valent functions. *Proceedings of the American Mathematical Society*, *46*(1), 29–37. https://doi.org/10.1090/S0002-9939-1974-0352440-1.

[27] K. I. Noor (1983). Hankel determinant problem for the class of functions with bounded boundary rotation. *Revue Roumaine de Mathématiques Pures et Appliquées*, *28*(8), 731–739. https://zbmath.org/0524.30008.

[28] C. Pommerenke (1975). *Univalent Functions. With a Chapter on Quadratic Differentials by Gerd Jensen* volume 25 of *Studia mathematica / Mathematische Lehrbücher*. Vandenhoeck & Ruprecht, Germany. https://zbmath.org/0298.30014.

[29] C. Pommerenke (1966). On the coefficients and Hankel determinants of univalent functions. *Journal of the London Mathematical Society*, *1*(1), 111–122. https://doi.org/10.1112/jlms/s1-41.1.111.

[30] C. Pommerenke (1967). On the Hankel determinants of univalent functions. *Mathematika*, *14*(1), 108–112. https://doi.org/10.1112/S002557930000807X.

[31] H. U. Rehman, M. Darus & J. Salah (2021). Generalizing certain analytic functions correlative to the n-th coefficient of certain class of bi-univalent functions. *Journal of Mathematics*, *2021*, Artical ID: 6621315. https://doi.org/10.1155/2021/6621315.

[32] F. M. Sakar & H. O. Güney (2017). Faber polynomial coefficient estimates for subclasses of m-fold symmetric bi-univalent functions defined by fractional derivative. *Malaysian Journal of Mathematical Sciences*, *11*(2), 275–287.

[33] G. Singh, G. Singh & G. Singh (2019). Certain subclasses of bi-close-to-convex functions associated with quasi-subordination. *Abstract and Applied Analysis*, *2019*, Article ID: 6947020. https://doi.org/10.1155/2019/6947020.

[34] H. M. Srivastava, G. Murugusundaramoorthy & S. M. El-Deeb (2021). Faber polynomial coefficient estimates of bi-close-to-convex functions connected with the Borel distribution of the Mittag-Leffler type. *Journal of Nonlinear and Variational Analysis*, *5*(1), 103–118. https://doi.org/10.23952/jnva.5.2021.1.07.

[35] H. M. Srivastava, A. K. Mishra & P. Gochhayat (2010). Certain subclasses of analytic and bi-univalent functions. *Applied Mathematics Letters*, *23*(10), 1188–1192. https://doi.org/10.1016/j.aml.2010.05.009.

[36] H. M. Srivastava, I. Al-Shbeil, Q. Xin, F. Tchier, S. Khan & S. N. Malik (2023). Faber polynomial coefficient estimates for bi-close-to-convex functions defined by the q-fractional derivative. *Axioms*, *12*(6), Article ID: 585. https://doi.org/10.3390/axioms12060585.

[37] H. Srivastava, S. S. Eker & R. M. Ali (2015). Coefficient bounds for a certain class of analytic and bi-univalent functions. *Filomat*, *29*(8), 1839–1843. https://doi.org/10.2298/FIL1508839S.

[38] D. Tan (1984). The coefficient estimates for bi-univalent functions. *Chinese Annals of Mathematics, Series A*, *5*, 559–568. https://zbmath.org/0548.30012.

[39] K. T. Wieclaw & P. Zaprawa (2018). On the coefficient problem for close-to-convex functions. *Turkish Journal of Mathematics*, *42*(5), 2809–2818. https://doi.org/10.3906/mat-1711-36.

[40] P. Zaprawa (2014). Estimates of initial coefficients for bi-univalent functions. In *Abstract and Applied Analysis*, volume 2014 pp. Article ID: 357480. Hindawi. https://doi.org/10.1155/2014/357480.

[41] J. Zhai, R. Srivastava & J. L. Liu (2022). Faber polynomial coefficient estimates of bi-close-to-convex functions associated with generalized hypergeometric functions. *Mathematics*, *10*(17), Article ID: 3073. https://doi.org/10.3390/math10173073.