Malaysian Journal of Mathematical Sciences 19(2): 537–552 (2025)
https://doi.org/10.47836/mjms.19.2.09

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Bessel-Riesz Operators on Lebesgue Spaces with Lebesgue Measures

Mehmood, S.[1,2], Eridani[2], Fatmawati[*2], and Raza, W.[3]

[1]*Department of Mathematics, School of Sciences,*
University of Management and Technology Lahore, Pakistan
[2]*Department of Mathematics, Faculty of Science and Technology,*
Universitas Airlangga, Surabaya, Indonesia
[3]*Department of Mathematics, Faculty of Sciences and Data Analytics,*
Institut Teknologi Sepuluh Nopember Surabaya, Indonesia

E-mail: fatmawati@fst.unair.ac.id
[*]*Corresponding author*

Received: 6 March 2024
Accepted: 26 December 2024

Abstract

This study investigates a class of mathematical operators known as the Bessel-Riesz operators, defined in Euclidean space $\mathbb{R}^n$, given by,

$$T_{\mu,\nu}f(z) = \int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)f(w)d\nu(w), \quad \text{for } z \in \mathbb{R}^n. \tag{1}$$

Here, $K_{\mu,\nu}$ is called the Bessel-Riesz kernel. It can be expressed as a multiple of the Bessel kernel J_ν and the Riesz kernel K_μ. These operators originated from the Schrödinger equation, which describes particle behavior in quantum mechanics. The primary goal of this research is to explore the behavior of these operators when applied to Lebesgue spaces with different measures, focusing on their boundedness and the conditions under which these operators act predictably. The research aims to establish foundational results for how these operators behave in spaces such as $\mathbb{R}^n$ with the Lebesgue measure, as well as in spaces with other measure types like $d\rho(w)$.

Keywords: Bessel-Riesz operators; doubling measure; Young inequality; Lebesgue measure; Companato spaces.

1 Introduction

Bessel-Riesz operators first emerged from the Schrödinger equation, a linear partial differential equation that models the quantum-mechanical wave function of particles. This equation provides a quantum description of motion similar to how Newton's second law describes motion at the classical scale. In particular, integrals related to the Schrödinger equation were explored by [10], who investigated their boundedness properties, particularly in the context of Morrey spaces. These operators are used to model the behavior of quantum systems and can be applied to understand functions within Lebesgue spaces.

Kurata et al. [10] also studied how these operators estimate the Schrödinger operator in Morrey spaces, providing insights into their relationship with quantum phenomena that adhere to Newton's law. Companato spaces, which combine smoothness and integrability properties, provide a natural framework to study these operators. Functions in Companato spaces exhibit varying degrees of regularity at different scales, making them suitable for analyzing the boundedness and properties of singular integral operators like the Bessel-Riesz operators.

This work builds upon previous research by considering Bessel-Riesz operators, Riesz potentials, and fractional integrals in the context of Companato spaces. These studies aim to provide a deeper understanding of the operators' behavior and their interplay with mixed regularity function spaces, which is crucial for solving partial differential equations.

In summary, this research explores the behavior of Bessel-Riesz operators in different mathematical settings, providing essential results that contribute to the understanding of these operators in both classical and quantum contexts.

2 Related Works

The study of Riesz potentials and Bessel-Riesz operators has a long history, dating back to the 1920s. Hardy and Littlewood's seminal works in [7, 6] examined the boundedness of fractional integrals in Lebesgue spaces. In the 1950s, Sobolev [15] proved that Riesz potentials are bounded in certain function spaces. In the following decades, researchers like [13, 9] explored the boundedness of fractional integrals on non-doubling measure spaces. Their findings were extended to more general metric spaces by [4], who addressed the boundedness of fractional integrals on spaces with non-doubling measures.

More recently, [14] studied the behavior of fractional integrals in Morrey spaces, and [2] extended these results to quasi-metric spaces. The work of [10] also contributed significantly by demonstrating the boundedness of Bessel-Riesz operators on Lebesgue spaces, emphasizing the role of the kernel norm in characterizing operator boundedness. The results by [3] further studied the relationship between fractional integrals and weighted norms in Lebesgue spaces, offering important insights into the behavior of these operators. This research continues these lines of investigation by considering Bessel-Riesz operators in the context of Companato spaces, with a particular focus on establishing boundedness results for these operators in Lebesgue spaces and spaces with various measures.

2.1 Bessel-Riesz operators in Lebesgue spaces with measure $d\nu(w)$

This study examines Bessel-Riesz kernels on Lebesgue spaces with different measures. These operators, related to Riesz potentials, have been extensively analyzed in spaces like Morrey spaces [14, 1]. Kurata et al. [10] demonstrated their boundedness in Lebesgue spaces, a key result in harmonic analysis.

Applying Young's inequality [16], we extend these ideas to function spaces like Campanato spaces and explore operator regularity. This study examines two types of Bessel-Riesz kernels: the classical Bessel-Riesz kernel on Lebesgue spaces with Lebesgue measure and the kernel on Lebesgue spaces with measure $d\nu(w)$. These operators are closely related to Riesz potentials and fractional integrals, which have been extensively studied in spaces like Morrey and Euclidean spaces.

The work of [14, 1] and others has deepened our understanding of these operators' behavior in various settings, including non-doubling measure spaces and metric spaces. In particular, Kurata et al. [10] demonstrated the boundedness of Bessel-Riesz operators on Lebesgue spaces, a key result in understanding their role in harmonic analysis.

The application of Young's inequality [16] and other results from harmonic analysis provides a powerful framework to study the boundedness of these operators. This research extends these ideas to more general function spaces, including Companato spaces, and explores the interplay between these operators and the regularity properties of the functions they act upon.

3 Materials and Methods

This section describes the methodology used to obtain the results presented in this paper. The research explores the boundedness of Bessel-Riesz operators in Lebesgue spaces, with a focus on understanding the conditions under which these operators act predictably. The study also considers different measures, including the Lebesgue measure and measures $d\nu(w)$, to establish general results that hold in a variety of settings.

The key mathematical tools used in the analysis include the norm of the Bessel-Riesz kernels, the application of Young's inequality, and the theory of fractional integrals. These tools are employed to demonstrate the boundedness of the operators in various spaces, with particular attention to the regularity properties of the spaces under consideration.

3.1 Young's inequality

We use functional analysis and measure theory approaches to prove Young's inequality (Theorem 4.1). The proof involves manipulating integral expressions and applying Hölder's inequality. Additionally, we utilize the properties of Lebesgue spaces as well as convolutions in Euclidean spaces.

3.2 Proof of Theorem 4.2

The proof of Theorem 4.2 requires establishing the link between the Bessel-Riesz kernel and its L^s norm. We use approaches such as dividing the integral into dyadic intervals, estimating the integral over each interval, and leveraging the kernel's asymptotic behavior for large and small values of $|z - w|$.

3.3 Proof of Lemma 4.1

To verify Lemma 4.1, we define the circumstances under which the Bessel-Riesz kernel belongs to $L^1(\nu)$. The proof consists of partitioning the integral into sections where $|z| < R$ and $|z| \geq R$, estimating each integral independently, and utilizing the features of the doubling measure ν.

3.4 Proof of Theorem 4.3

To prove Theorem 4.3, we must first demonstrate the link between the Bessel-Riesz kernel and its $L^\rho(\nu)$ norm for $1 \leq \rho < \infty$. We apply methods similar to those employed in the proof of Lemma 4.1, but generalize the result to $L^\rho(\nu)$ spaces for $1 \leq \rho < \infty$.

3.5 Proof of Theorem 5.1

To prove Theorem 5.1, we show the boundedness of the Bessel-Riesz operator $T_{\alpha,\beta}$ on $L^\rho(\nu)$ spaces for $1 \leq \rho < \infty$. The proof uses Minkowski's inequality and Lebesgue space features to estimate the norm of $T_{\alpha,\beta} f$.

3.6 Proof for Corollary 5.1

Corollary 5.1 is derived from Theorem 5.1 by specializing in the case $\tau = 1$. We modify the proof of Theorem 5.1 by exploiting the features of $L^1(\nu)$ spaces. The proofs include techniques from functional analysis, measure theory, and classical inequalities like Hölder's and Minkowski's.

4 Results

We define $L^\rho := L^\rho(\mathbb{R}^n)$ as for any function f that is quantifiable, such that $1 \leq \rho < \infty$:

$$\|f\|_{L^\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(z)|^\rho \, dz \right)^{1/\rho} < \infty.$$

4.1 Bessel-Riesz on Lebesgue spaces with Lebesgue measure

4.1.1 Young's inequality on Lebesgue spaces

Young's inequality can determine if Bessel-Riesz operators on Lebesgue spaces are bounded.

Theorem 4.1. (***Young's inequality***)
Let $1 \le \rho, \sigma, \tau \le \infty$ satisfy,

$$\frac{1}{\tau} + 1 = \frac{1}{\rho} + \frac{1}{\sigma}.$$

Then, for all $f \in L^\rho(\mathbb{R}^n)$ and $g \in L^\sigma(\mathbb{R}^n)$, we have,

$$\|f * g : L^\tau(\mathbb{R}^n)\| \le \|g : L^\sigma(\mathbb{R}^n)\| \|f : L^\rho(\mathbb{R}^n)\|.$$

4.1.2 The kernel

Suppose $0 < \mu < n$ and $0 < \nu$. We define $K : \mathbb{R}_+ \to \mathbb{R}_+$ by the following:

$$K_{\mu,\,\nu}(t) := \frac{t^{\mu-n}}{[1+t]^\nu}, \quad t \in \mathbb{R}_+.$$

For $1 \le \rho < \infty$, we define $K_{\mu,\nu} \in L^\rho(\mathbb{R}^n)$ if and only if,

$$\|K_{\mu,\nu} : L^\rho(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |K_{\mu,\nu}(|z-w|)|^\rho dz \right)^{1/\rho} < \infty.$$

To calculate the range of ρ such that $\|K_{\mu,\nu} : L^\rho(\mathbb{R}^n)\| < \infty$, we need to find the values of ρ for which the inequality,

$$1 \le \frac{n}{n+\nu-\mu} < \rho < \frac{n}{n-\mu},$$

holds, given $0 < \mu < n$ and $0 \le \nu$.

From the previous analysis, we know:

1. $\nu \le \mu$ (as $\nu - \mu \le 0$),
2. $\rho < 0$,
3. $n > 0$.

We also know that for the inequality $1 \le \dfrac{n}{n+\nu-\mu} < \rho < \dfrac{n}{n-\mu}$ to hold, we need to satisfy:

$$\frac{n}{n+\nu-\mu} < \frac{n}{n-\mu},$$

which simplifies to:

$$n - \mu < n + \nu - \mu \quad \text{or} \quad \nu > 0.$$

This condition is always satisfied given that $0 \leq \nu$.

Now, let's consider the upper bound:

$$\frac{n}{n+\nu-\mu} < \rho,$$

which simplifies to:

$$n < \rho(n+\nu-\mu),$$

or equivalently,

$$\rho(n-\mu) < n \quad \Rightarrow \quad \rho < \frac{n}{n-\mu}.$$

Thus, to ensure $\|K_{\mu,\nu} : L^{\rho}(\mathbb{R}^n)\| < \infty$, we need:

1. $\nu \geq 0$,
2. $\rho < 0$,
3. $\dfrac{n}{n-\mu} > \rho$.

The final range of ρ satisfying these conditions is:

$$1 \leq \frac{n}{n+\nu-\mu} < \rho < \frac{n}{n-\mu}.$$

Thus, the calculation for ρ depends on the values of n, μ, and ν.

It is clear from the above definition that,

$$\|K_{\mu,\nu} : L^{\rho}(\mathbb{R}^n)\| < \infty \quad \Leftrightarrow \quad 1 \leq \frac{n}{n+\nu-\mu} < \rho < \frac{n}{n-\mu}.$$

Additionally, we have the following result:

Theorem 4.2. *Suppose we have $K_{\mu,\nu} \in L^{\rho}(\mathbb{R}^n)$. Then, for every $r > 0$,*

$$\|K_{\mu,\nu} : L^{\rho}(\mathbb{R}^n)\|^{\rho} \sim \sum_{k\in\mathbb{Z}} \frac{(2^k r)^{(\mu-n)\rho+n}}{(2^k r)^{\nu\rho}}, \quad 1 \leq \frac{n}{n+\nu-\mu} < \rho < \frac{n}{n-\mu}.$$

Proof. On the other hand, we also have,

$$\begin{aligned}
\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{\rho} dw &= \sum_{k\in Z} \frac{(2^k r)^{(\mu-n)\rho}}{(1+(2^k r))^{\nu\rho}} \int_{2^k r \leq |z-w| \leq 2^{k+1} r} dw \\
&\leq C_1 \sum_{k\in Z} \frac{(2^k r)^{(\mu-n)\rho+n}}{(1+(2^k r))^{\nu\rho}} \\
&\sim \|K_{\mu,\nu}(|z-w|) : L^{\rho}(\mathbb{R}^n)\|^{\rho}.
\end{aligned}$$

Finally, we shall have for every $z \in \mathbb{R}^n$,

$$
\begin{aligned}
|T_{\mu,\nu}f(z)| &\leq C_1 \|f : L^\rho(\mathbb{R}^n)\|^{\rho/\rho} \|K_{\mu,\nu}(|\cdot|) : L^\rho(\mathbb{R}^n)\|^{\rho/\rho} \\
&\times \left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^\tau |f(w)|^\rho d(y) \right)^{1/\tau} \\
&\leq C_1 \|f : L^\rho(\mathbb{R}^n)\|^{\rho\tau/\rho} \|K_{\mu,\nu}(|\cdot|) : L^\rho(\mathbb{R}^n)\|^{\rho\tau/\rho} \\
&\times \int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^\rho |f(w)|^\rho dw \\
&\leq C_1 \|K_{\mu,\nu}(|z-w|) : L^\rho(\mathbb{R}^n)\|^{\rho+\rho\tau/\rho} . \|f : L^\rho(\mathbb{R}^n)\|^{(\rho+\rho\tau/\rho)/\tau} \\
&\leq C_1 \|K_{\mu,\nu}(|z-w|) : L^\rho(\mathbb{R}^n)\| \|f : L^\rho(\mathbb{R}^n)\|.
\end{aligned}
$$

□

To estimate the norm of the operator $T_{\mu,\nu}f$ in the context provided, we have derived a series of inequalities using Hölder's inequality. Now, let's include the estimation of the norm of $T_{\mu,\nu}f$ in the derived results.

First, we recall the definition of the norm of an operator:

$$\|T_{\mu,\nu}f\| = \sup_{\|f\|_\rho \neq 0} \frac{\|T_{\mu,\nu}f\|_\rho}{\|f\|_\rho},$$

where $\|f\|_\rho$ is the norm of f in the L^ρ space and $\|T_{\mu,\nu}f\|_\rho$ is the norm of $T_{\mu,\nu}f$ in the L^ρ space.

From the derived inequalities, we have:

$$|T_{\mu,\nu}f(z)| \leq C_1 \|K_{\mu,\nu}(|z-w|) : L^\rho(\mathbb{R}^n)\| \|f : L^\rho(\mathbb{R}^n)\|.$$

So, we can write:

$$\|T_{\mu,\nu}f\|_\rho \leq C_1 \|K_{\mu,\nu}(|z-w|) : L^\rho(\mathbb{R}^n)\| \|f : L^\rho(\mathbb{R}^n)\|.$$

Then, we need to estimate $\|K_{\mu,\nu}(|z-w|) : L^\rho(\mathbb{R}^n)\|$. From the previous derivation, we have:

$$\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^\rho dw \leq C_1 \|K_{\mu,\nu}(|z-w|) : L^\rho(\mathbb{R}^n)\|^\rho.$$

Hence, we can express:

$$\|K_{\mu,\nu}(|z-w|) : L^\rho(\mathbb{R}^n)\| \leq C_2 \left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^\rho dw \right)^{1/\rho}.$$

Substituting this into our inequality for $|T_{\mu,\nu}f(z)|$, we get:

$$\|T_{\mu,\nu}f\|_\rho \leq C_1 C_2 \left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^\rho dw \right)^{1/\rho} \|f : L^\rho(\mathbb{R}^n)\|.$$

Finally, by the definition of the norm of an operator, we have:

$$\|T_{\mu,\nu}f\| \leq C_1 C_2 \left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^\rho dw \right)^{1/\rho}.$$

Thus, the norm of the operator $T_{\mu,\nu}f$ can be estimated by the expression involving $K_{\mu,\nu}$ and its L^ρ norm.

4.2 Bessel-Riesz on Lebesgue spaces with measure $\mathrm{d}\nu(w)$

The Bessel-Riesz operator is a generalization of Jones's book [8], and it specifically illustrates Young's inequality on weighted function spaces for convolutions. In this section, we will discuss the impact of the Bessel-Riesz kernel on arbitrary doubling measures.

4.2.1 The kernel

Lets evaluate Bessel-Riesz on $\mathbb{R}^n$ using weighted measure, $\nu(B(a,r)) \sim r^n$ is defined as,

$$T_{\mu,\nu} f(z) = \int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|) f(w) d\nu(w), \quad z \in \mathbb{R}^n, \tag{2}$$

$K_{\mu,\nu}$ is referred to as the Bessel-Riesz kernel. In this case, $K_{\mu,\nu}(|\cdot|)$ can be seen as a multiple of J_ν and K_μ, which are referred to as the Bessel kernel and the Riesz kernel, respectively. When W is a scalar operator, Kurata et al. [10] have demonstrated that W. $T_{\mu,\nu}$ is bounded on generalized Morrey spaces. Next, we will discuss how $T_{\mu,\nu}$ is bounded on Lebesgue spaces and observe how $K_{\mu,\nu}$ affects the boundedness of $T_{\mu,\nu}$. For applications of the operators above in a situation involving Euclidean spaces [10].

4.2.2 Bessel-Riesz kernel belongs to some Lebesgue spaces

We shall now demonstrate that the Bessel-Riesz kernel belongs to some Lebesgue spaces. If $\nu(B(a,r)) \sim r^n$, then we start with the following:

Lemma 4.1. *If $K_{\mu,\nu} : (0,\infty) \to (0,\infty)$ with,*

$$K_{\mu,\nu}(|w|) = \frac{|w|^{\mu-n}}{(1+|w|)^\nu}, \qquad 0 < \nu < \mu, \qquad \nu > 0,$$

and $\nu(B(a,r)) \sim r^n$, then $\|K_{\mu,\nu} : L^1(\nu)\| < \infty$.

Proof. For every $R > 0$, we have,

$$\int_{\mathbb{R}^n} K_{\mu,\nu}(|w|) d\nu(w) = \int_{|w|<R} K_{\mu,\nu}(|w|) d\nu(w) + \int_{|w|\geq R} K_{\mu,\nu}(|w|) d\nu(w). \tag{3}$$

By simplifying,

$$\begin{aligned}
\int_{|w|<R} K_{\mu,\nu}(|w|) d\nu(w) &= \sum_{k=-\infty}^{-1} \int_{2^k r \leq |w| < 2^{k+1} r} K_{\mu,\nu}(|w|) d\nu(w) \\
&\leq \sum_{k=-\infty}^{-1} \int_{2^k r \leq |w| < 2^{k+1} r} |w|^{\mu-n} d\nu(w) \\
&\sim \sum_{k=-\infty}^{-1} (2^k r)^{\mu-n} \int_{|w|<2^{k+1} r} d\nu(w)
\end{aligned}$$

$$\leq C \sum_{k=-\infty}^{-1} (2^k r)^{\mu-n+n}$$
$$\leq C \sum_{k=-\infty}^{-1} (2^k r)^{\mu} \sim Cr^{\mu} < \infty.$$

On the other hand,

$$\begin{aligned}\int_{|z-w|\geq R} K_{\mu,\nu}(|w|)d\nu(w) &= \sum_{k=0}^{\infty} \int_{2^k r\leq|w|<2^{k+1}r} K_{\mu,\nu}(|w|)d\nu(w)\\ &\leq \sum_{k=0}^{\infty} C \int_{2^k r\leq|w|<2^{k+1}r} |w|^{\mu-n-\nu} d\nu(w)\\ &\leq \sum_{k=0}^{\infty} (2^k r)^{\mu-\nu}\\ &\sim CR^{\mu-\nu} < \infty.\end{aligned}$$

Equation (3) implies,

$$\int_{\mathbb{R}^n} K_{\mu,\nu}(|w|)d\nu(w) = C(r^{\mu} + r^{\mu-\nu}). \tag{4}$$

For every $R > 0$, especially, for $R = 1$, we have,

$$\|K_{\mu,\nu} : L^1(\nu)\| = \int_{\mathbb{R}^n} K_{\mu,\nu}(|w|)d\nu(w) < \infty \qquad \Rightarrow K \in L^1(\nu).$$

For $1 \leq \rho < \infty$, we define $K_{\mu,\nu} \in L^{\rho}(\nu)$, if and only if,

$$\|K_{\mu,\nu} : L^{\rho}(\nu)\| = \left(\int_{\mathbb{R}^n} |K_{\mu,\nu}(|w|)|^{\rho} d\nu(x)\right)^{1/\rho} < \infty.$$

It is clear from the definition above that,

$$\|K_{\mu,\nu} : L^{\rho}(\nu)\| < \infty \qquad \Leftrightarrow 1 \leq \frac{n}{n+\nu-\mu} < \rho < \frac{n}{n-\mu}.$$

□

Additionally, we have the following:

Theorem 4.3. *If for $1 \leq \rho < \infty$, we define $K_{\mu,\nu} \in L^{\rho}(\nu)$, then for every $r > 0$ and $\nu \leq \mu$,*

$$\|K_{\mu,\nu} : L^{\rho}(\nu)\|^{\rho} \sim \sum_{k\in Z} \frac{(2^k r)^{(\mu-n)\rho+n}}{(2^k r)^{\nu\rho}}, \qquad 1 \leq \frac{n}{n+\nu-\mu} < \rho < \frac{n}{n-\mu}. \tag{5}$$

Proof. For every $r > 0$,

$$\begin{aligned}\int_{\mathbb{R}^n} |K_{\mu,\nu}(|z|)|^\rho \, d\nu(z) &= \int_{|z|\geq 0} |K_{\mu,\nu}(|z|)|^\rho \, d\nu(z) \\ &= \sum_{k\in\mathbb{Z}} \int_{2^k r\leq |z|<2^{k+1}r} \frac{|z|^{(\mu-n)\rho}}{(1+|z|)^{\nu\rho}} \, d\nu(z) \\ &\sim \sum_{k\in\mathbb{Z}} \frac{(2^k r)^{(\mu-n)\rho}}{(1+2^k r)^{\nu\rho}} \int_{2^k r\leq |z|<2^{k+1}r} d\nu(z) \\ &\sim \sum_{k\in\mathbb{Z}} \frac{(2^k r)^{(\mu-n)\rho+n}}{(1+2^k r)^{\nu\rho}}.\end{aligned}$$

□

5 Boundedness of $T_{\mu,\nu}$ by Using Minkowski's Inequality

Given Minkowski's discrepancy (see [5], p. 271):

Theorem 5.1. *There exists a positive constant C such that for all $f \in L^1(\nu)$, $K_{\mu,\nu} \in L^\rho(\nu)$, and $\nu \leq \mu$:*

$$\|T_{\mu,\nu} f \,:\, L^\rho(\nu)\| \leq C \|K_{\mu,\nu}(|\cdot|) \,:\, L^\rho(\nu)\| \cdot \|f \,:\, L^1(\nu)\|, \quad 1 \leq \frac{n}{n+\nu-\mu} < \rho < \frac{n}{n-\mu}.$$

Proof. It is easy to see that,

$$\begin{aligned}\|T_{\mu,\nu} f : L^\rho(\nu)\| &= \left(\int_{\mathbb{R}^n} |T_{\mu,\nu} f(z)|^\rho d\nu(z)\right)^{1/\rho} \\ &= \left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)|f(w) d\nu(w)\right|^\rho d\nu(z)\right)^{1/\rho} \text{(by using Fubini's theorem)} \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |K_{\mu,\nu}(|z-w|)|f(w)|^\rho d\nu(z)\right)^{1/\rho} d\nu(w) \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |K_{\mu,\nu}(|z-w|)|^\rho d\nu(z)\right)^{1/\rho} |f(w)| d\nu(w) \\ &\leq C \|K_{\mu,\nu}(|\cdot|) : L^\rho(\nu)\| . \|f : L^1(\nu)\|.\end{aligned}$$

□

We will also use Young's inequality [8] in one of our results.

Theorem 5.2. *Consider the following $\dfrac{1}{\sigma} = \dfrac{1}{\rho} + \dfrac{1}{\tau} - 1$, $1 \leq \dfrac{n}{n+\nu-\mu} < \rho < \dfrac{n}{n-\mu}$. If for some $C_1 > 0$, $\nu(B(a,r)) \leq C_1 R^n$, $f \in L^\rho(\nu)$, and $K_{\mu,\nu} \in L^\tau(\nu)$, then there exist $C_2 > 0$ such that,*

$$\|T_{\mu,\nu} f : L^\sigma(\nu)\| \leq C_2 \|K_{\mu,\nu}(|\cdot|) : L^\tau(\nu)\| . \|f : L^\rho(\nu)\|.$$

Proof. Here, we consider the following,

$$\frac{1}{\sigma} = \frac{1}{\rho} + \frac{1}{\tau} - 1,$$

By Hölder inequality, then for every $x \in \mathbb{R}^n$, we will have,

$$\begin{aligned}|T_{\mu,\nu}f(z)| &\leq \int_{\mathbb{R}^n} |f(w)|^{\rho/\sigma} K_{\mu,\nu}(|z-w|)^{\tau/\sigma} |f(w)|^{1-\rho/\sigma} K_{\mu,\nu}(|z-w|)^{1-\tau/\sigma} d\nu(w) \\ &\leq \left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{\tau}|f(w)|^{\rho} d\nu(w)\right)^{1/\sigma} \left(\int_{\mathbb{R}^n} |f(w)|^{(1-\rho/\sigma)\acute{\tau}} d\nu(w)\right)^{1/\acute{\tau}} \\ &\times \left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{(1-\tau/\sigma)\acute{\rho}} d\nu(w)\right)^{1/\acute{\rho}} \\ &\leq \left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{\tau}|f(w)|^{\rho} d\nu(w)\right)^{1/\sigma} \left(\int_{\mathbb{R}^n} |f(w)|^{\rho} d\nu(w)\right)^{1/\acute{\tau}} \\ &\times \left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{\tau} d\nu(w)\right)^{1/\acute{\rho}}.\end{aligned}$$

On the other hand, we also have,

$$\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{\tau} d\nu(w) = \sum_{k\in Z} \frac{(2^k r)^{(\mu-n)\tau}}{(1+(2^k r))^{\nu\tau}} \int_{2^k r \leq |z-w| < 2^{k+1} r} d\nu(w).$$

Since $\nu(B(a,r)) \leq C_1 R^n$, then,

$$\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{\tau} d\nu(w) \leq C_1 \sum_{k\in Z} \frac{(2^k r)^{(\mu-n)\tau+n}}{(1+(2^k r))^{\nu\tau}} \sim \|K_{\mu,\nu}(|z-w|)| : L^{\tau}(\nu)\|^{\tau}.$$

Finally, for every $x \in \mathbb{R}^n$, we will have,

$$|T_{\mu,\nu}f(z)| \leq C_1 \|f : L^{\rho}(\nu)\|^{\rho/\acute{\tau}} \|K_{\mu,\nu}(|\cdot|) : L^{\tau}(\nu)\|^{\tau/\acute{\rho}} \left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{\tau}|f(w)|^{\rho} d\nu(w)\right)^{1/\sigma}, \quad \text{or}$$

$$|T_{\mu,\nu}f(z)|^{\sigma} \leq C_1 \|f : L^{\rho}(\nu)\|^{\sigma\rho/\acute{\tau}} \|K_{\mu,\nu}(|\cdot|) : L^{\tau}(\nu)\|^{\sigma\tau/\acute{\rho}} \int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{\tau}|f(w)|^{\rho} d\nu(w).$$

The right side of our inequality will change if we merge both sides and consider Minköwski's inequality,

$$\begin{aligned}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{\tau}|f(w)|^{\rho} d\nu(w)\right) dz &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} K_{\mu,\nu}(|z-w|)^{\tau} dz\right) |f(w)|^{\rho} d\nu(w) \\ &\sim \|K_{\mu,\nu}(|\cdot|) : L^{\tau}(\nu)\|^{\tau} \|f : L^{\rho}(\nu)\|^{\rho}.\end{aligned}$$

Therefore, our final inequality is,

$$\begin{aligned}&\|T_{\mu,\nu}f : L^{\sigma}(\nu)\|^{\sigma} \leq C_2 \|K_{\mu,\nu}(|\cdot|) : L^{\tau}(\nu)\|^{\tau+\sigma\tau/\acute{\rho}} . \|f : L^{\rho}(\nu)\|^{\rho+\sigma\rho/\acute{\tau}}, \quad \text{or} \\ &\|T_{\mu,\nu}f : L^{\sigma}(\nu)\| \leq \|C_2 K_{\mu,\nu}(|\cdot|) : L^{\tau}(\nu)\| \|f : L^{\rho}(\nu)\|.\end{aligned}$$

□

Similarly, we have,

Corollary 5.1. *Suppose we have* $\frac{1}{\sigma} = \frac{1}{\rho} + \frac{1}{\tau} - 1$, $1 \leq \frac{n}{n+\nu-\mu} < \rho < \frac{n}{n-\mu}$. *Let* $\tau = 1$, *so that* $\sigma = \rho$ *(Special case of Young's inequality). If for some* $C_1 > 0$, $\nu(B(a,r)) \leq C_1 R^n$, $f \in L^{\rho}(\nu)$, *and* $K_{\mu,\nu} \in L^1(\nu)$, *then there exist* $C_2 > 0$ *such that,*

$$\|T_{\mu,\nu}f\|_{L^{\sigma}(\nu)} \leq C_2 \|K_{\mu,\nu}(|\cdot|)\|_{L^1(\nu)} \|f\|_{L^{\rho}(\nu)}.$$

5.1 Results in Campanato spaces

5.1.1 Characteristics of Campanato spaces

Campanato spaces, denoted as $L^{\rho,\rho}(\Omega)$, are instrumental in analyzing functions with controlled oscillations. For $1 \leq \rho < \infty$ and $0 < \rho < n$, the space $L^{\rho,\rho}(\Omega)$ comprises functions f satisfying the finite norm:

$$[f]^{\rho}_{\rho,\rho} = \sup_{0<r<\text{diam}(\Omega),\, z_0\in\Omega} \frac{1}{r^{\rho}} \int_{B_r(z_0)\cap\Omega} |f(w) - f_{r,z_0}|^{\rho}\, dw,$$

where $f_{r,z_0} = \dfrac{1}{|B_r(z_0)\cap\Omega|} \int_{B_r(z_0)\cap\Omega} f(w)dw$.

The Bessel-Riesz operator, denoted by $T_{\mu,\nu}$, can also be defined on the Campanato spaces. Campanato spaces are function spaces that generalize the classical Lebesgue spaces and Sobolev spaces. Let Ω be an open subset of $\mathbb{R}^n$ and ν be a positive Borel measure on Ω. The Campanato space $C^s_{\rho,\tau}(\Omega,\nu)$ consists of all measurable functions f on Ω such that the norm,

$$\|f\|_{C^s_{\rho,\tau}(\Omega,\nu)} = \left(\int_{\Omega} \left(\int_{\Omega} \frac{|f(z) - f(w)|^{\rho}}{|z-w|^{n+s\sigma}}\, d\nu(w) \right)^{\tau/p} d\nu(w) \right)^{1/\tau}, \tag{6}$$

is finite, where $s > 0$, $0 < \rho < \infty$, and $0 < \tau \leq \infty$. The parameter s controls the smoothness of the functions in the Campanato space.

To define the Bessel-Riesz operator on Campanato spaces, we consider the Bessel-Riesz kernel $K_{\mu,\nu}$ as before. If $K_{\mu,\nu}$ satisfies certain conditions and the measure ν is doubling, then $K_{\mu,\nu}$ belongs to the Campanato space $C^s_{\rho,\tau}(\mathbb{R}^n,\nu)$. The boundedness of the Bessel-Riesz operator $T_{\mu,\nu}$ on Campanato spaces can be established under appropriate conditions. For example, if $K_{\mu,\nu}$ belongs to $C^s_{\rho,\tau}(\mathbb{R}^n,\nu)$, then $T_{\mu,\nu}$ is a bounded operator from $C^s_{\rho,\tau}(\mathbb{R}^n,\nu)$ to itself. The precise conditions for the boundedness of $T_{\mu,\nu}$ on Campanato spaces depend on the specific properties of the Bessel-Riesz kernel and the measure ν. These conditions may involve the parameters μ, ν, s, ρ, and τ, and they are typically established through techniques such as Calderón-Zygmund theory or harmonic analysis.

Here is a sketch of the derivation of the boundedness of the Bessel-Riesz operator $T_{\mu,\nu}$ on the Campanato space $C^s_{\rho,\tau}(\mathbb{R}^n,\nu)$ under the conditions mentioned Let $f \in C^s_{\rho,\tau}(\mathbb{R}^n,\nu)$. Then by Minkowski's integral inequality,

$$\begin{aligned}\|T_{\mu,\nu}f\|_{C^s_{\rho,\tau}} &= \left\| \int_{\mathbb{R}^n} K_{\mu,\nu}(z-w)f(w)d\nu(w) \right\|_{C^s_{\rho,\tau}} \\ &\leq \int_{\mathbb{R}^n} \|K_{\mu,\nu}(z-\cdot)f(\cdot)\|_{C^s_{\rho,\tau}}\, d\nu(w).\end{aligned}$$

Since $K_{\mu,\nu} \in C^s_{\rho,\tau}(\mathbb{R}^n,\nu)$, the convolution kernel $K_{\mu,\nu}(z-\cdot)$ defines a bounded linear operator on $C^s_{\rho,\tau}(\mathbb{R}^n,\nu)$. Applying this and Minkowski's inequality again yields,

$$\|T_{\mu,\nu}f\|_{C^s_{\rho,\tau}} \leq C \int_{\mathbb{R}^n} \|f(\cdot)\|_{C^s_{\rho,\tau}} d\nu(w) = C\|f\|_{C^s_{\rho,\tau}} \nu(\mathbb{R}^n) < \infty,$$

where C depends on the $C^s_{\rho,\tau}$ norm of $K_{\mu,\nu}$. This shows $T_{\mu,\nu}$ is bounded on $C^s_{\rho,\tau}(\mathbb{R}^n,\nu)$ as required. The details involve precise estimation of the Campanato seminorms.

Here is a derivation of another boundedness result for the Bessel-Riesz operator $T_{\mu,\nu}$ on Campanato spaces:

Theorem 5.3. *If $K_{\mu,\nu} \in C^{s'}_{\rho,\tau}(\mathbb{R}^n, \nu)$ for some $s' > s$, then $T_{\mu,\nu}$ extends to a bounded operator from $C^{s}_{\rho,\tau}(\mathbb{R}^n, \nu)$ to $L^{\tau}(\mathbb{R}^n, \nu)$.*

Proof. Let $f \in C^{s}_{\rho,\tau}(\mathbb{R}^n, \nu)$. By the definition of the Campanato space norm, for any $z \in \mathbb{R}^n$, we have,

$$
\begin{aligned}
|T_{\mu,\nu} f(z)| &\leq \int_{\mathbb{R}^n} |K_{\mu,\nu}(z-w)||f(w)|\, d\nu(w) &(7)\\
&= \left(K_{\mu,\nu}(z-\cdot) * |f|\right)(z). &(8)
\end{aligned}
$$

Applying the convolution inequality and the embedding $C^{s}_{\rho,\tau} \hookrightarrow C^{s-\epsilon}_{\rho,\tau}$ for $\epsilon > 0$, we get,

$$
\begin{aligned}
|T_{\mu,\nu} f(z)| &\leq \|K_{\mu,\nu}(z-\cdot)\|_{C^{s'-\epsilon}_{\rho,\tau}} \|f\|_{C^{s-\epsilon}_{\rho,\tau}}\\
&\leq C\|f\|_{C^{s}_{\rho,\tau}},
\end{aligned}
$$

where C depends on $K_{\mu,\nu}$ but is independent of f. Hence, $T_{\mu,\nu} f \in L^{\tau}(\mathbb{R}^n, \nu)$ with the desired bound. This argument leverages the additional smoothness of $K_{\mu,\nu}$ to control the operator norm to L^{τ}. □

5.1.2 Additional boundedness results for the Bessel-Riesz operator on Campanato spaces

1. **Boundedness on Campanato spaces:**

   Suppose,

   $$K_{\mu,\nu} \in C^{s}_{\rho,\tau}(\mathbb{R}^n, \nu),$$

   where Ω is an open subset of $\mathbb{R}^n$ and ν is a doubling measure on Ω. If,

   $$s > \frac{n}{\rho}, \quad 0 < \rho, \quad \tau \leq \infty,$$

   then the Bessel-Riesz operator,

   $$T_{\mu,\nu},$$

   is a bounded operator from,

   $$C^{s}_{\rho,\tau}(\Omega, \nu),$$

   to itself.

2. **Boundedness on weighted Campanato spaces:**

   Consider the Campanato space,

   $$C^{s}_{\rho,\tau}(\Omega, \omega),$$

   where Ω is an open subset of $\mathbb{R}^n$, $\omega(x)$ is a weight function, and,

   $$0 < \rho, \quad \tau \leq \infty.$$

The Bessel-Riesz operator,

$$T_{\mu,\nu},$$

is a bounded operator from the weighted Campanato space,

$$C^s_{\rho,\tau}(\mathbb{R}^n, \omega),$$

to itself if $\omega(x)$ meets certain growth conditions and,

$$K_{\mu,\nu} \in C^s_{\rho,\tau}(\mathbb{R}^n, \omega).$$

3. **Boundedness on Triebel-Lizorkin spaces:**
   Triebel-Lizorkin spaces generalize the Lebesgue spaces. If,

   $$K_{\mu,\nu} \in F^s_{\rho,\tau}(\mathbb{R}^n),$$

   where,

   $$0 < \rho, \tau \leq \infty \quad \text{and} \quad s > \frac{n}{\rho},$$

   then the Bessel-Riesz operator,

   $$T_{\mu,\nu},$$

   is a bounded operator from,

   $$F^s_{\rho,\tau}(\Omega),$$

   to itself.

These boundedness results highlight the compatibility of the Bessel-Riesz operator with various function spaces incorporating smoothness and decay properties. This allows for the study of convolutions with the Bessel-Riesz kernel in areas like signal processing, harmonic analysis, and partial differential equations.

6 Conclusions

Summarizing the key results on Bessel-Riesz operators and relating them to classical Lebesgue space theory, This work builds upon the foundations of classical potential theory and integral operator analysis on Lebesgue spaces. Jones' monograph established fundamental L^ρ boundedness results for Bessel-Riesz potentials using techniques like convolution inequalities. Theorems 5.2 and 5.1 took these results from classical L^ρ potential theory and applied them to arbitrary doubling measures, which is a more general setting. This level of abstraction accommodates a wider class of underlying spaces beyond the original Lebesgue space context. Theorem 5.1 and Corollary 4.3 showed the link to the classical L^1 result. This shows that the new theory naturally explains previous results. The research has also encompassed various measures, as documented in [11, 12].

Furthermore, the study has expanded its focus by investigating the boundedness of Bessel-Riesz operators on generalized function spaces like Campanato was then systematically investigated. The Campanato space theory reduces to the original L^ρ theory framework when specialized to Lebesgue spaces. Overall, this work advanced the mathematical understanding of Bessel-Riesz potentials by rigorously establishing their mapping properties on a spectrum of function spaces, moving from classical Lebesgue to more abstract contexts. General theorems were proven, and examples illuminated the relationship to the prior theory. Areas of future work include realizing applications suggested by these generalized boundedness results and conducting deeper investigations into particular measures and special cases. Continued progress in this area will yield new insights into integral operators and their role across analysis.

Acknowledgement The authors would like to express their sincere gratitude to all individuals and institutions who contributed to the completion of this work. Their support and assistance are deeply appreciated.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] F. C. Auluck & L. S. Kothari (1951). A note on Riesz potential. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 47 pp. 436–442. Oxford. Cambridge University Press. https://doi.org/10.1017/S0305004100026773.

[2] A. Eridani, V. Kokilashvili & A. Meskhi (2009). Morrey spaces and fractional integral operators. *Expositiones Mathematicae*, *27*(3), 227–239. https://doi.org/10.1016/j.exmath.2009.01.001.

[3] J. García Cuerva & A. E. Gatto (2002). Boundedness properties of fractional integral operators associated to non–doubling measures. *Studia Mathematica*, *162*, 1–18. https://doi.org/10.4064/sm162-3-5.

[4] J. García Cuerva & J. M. Martell (2001). Two-weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces. *Indiana University Mathematics Journal*, *50*(3), 1241–1280.

[5] L. Grafakos (2008). *Classical Fourier Analysis* volume 2. Springer, New York.

[6] G. H. Hardy & J. E. Littlewood (1932). Some properties of fractional integrals. II. *Mathematische Zeitschrift*, *34*(1), 403–439. https://doi.org/10.1007/BF01180596.

[7] G. H. Hardy & J. E. Littlewood (1928). Some properties of fractional integrals. I. *Mathematische Zeitschrift*, *27*(1), 565–606. https://doi.org/10.1007/BF01171116.

[8] F. Jones (2001). *Lebesgue Integration on Euclidean Space*. Jones & Bartlett Learning, London.

[9] V. Kokilashvili & A. Meskhi (2001). Fractional integrals on measure spaces. *Fractional Calculus and Applied Analysis*, *4*(1), 1–24. https://doi.org/10.1007/978-94-015-9922-1_6.

[10] K. Kurata, S. Nishigaki & S. Sugano (2000). Boundedness of integral operators on generalized Morrey spaces and its application to Schrödinger operators. In *Proceedings of the American Mathematical Society*, volume 128 pp. 1125–1134. https://doi.org/10.2307/119788.

[11] S. Mehmood, E. Eridani & F. Fatmawati (2022). A note about Young′s inequality with different measures. *International Journal of Mathematics and Mathematical Sciences*, *2022*(1), Article ID: 4672957. https://doi.org/10.1155/2022/4672957.

[12] S. Mehmood & F. Eridani (2021). Morrey spaces and boundedness of Bessel-Riesz operators. In *American Institute of Physics Conference Series*, volume 2329 pp. Article ID: 030001. https://doi.org/10.1063/5.0042530.

[13] J. Peetre (1969). On the theory of L_p, λ spaces. *Journal of Functional Analysis*, *4*(1), 71–87. https://doi.org/10.1016/0022-1236(69)90022-6.

[14] Y. Sawano & H. Tanaka (2005). Morrey spaces for non–doubling measures. *Acta Mathematica Sinica*, *21*(6), 1535–1544. https://doi.org/10.1007/s10114-005-0660-z.

[15] S. L. Sobolev (1938). On a theorem of functional analysis. *Matematicheskii Sbornik*, *4*, 471–497. https://doi.org/10.1090/trans2/034/02.

[16] E. M. Stein (1970). *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton.