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On Exact Sequences for Some Evolution Algebras

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Abstract

In this paper, we construct exact sequences within evolution algebras, with a focus on their structure and properties. We provide a framework for building exact sequences and include illustrative examples. Key results include the isomorphism theorem, the splitting lemma, and the Five Lemma. Furthermore, we apply these sequences to the class of $S-$evolution algebras.

Keywords: evolution algebras; exact sequences; algebraic structures; isomorphism theorem; splitting lemma; five lemma.

1 Introduction

In Tian and Piotr [16], evolution algebras were introduced as a model for non-Mendelian genetics. Later, Tian [15] published a monograph that establishing numerous connections between evolution algebras and other areas of mathematics, such as graph theory, stochastic processes, group theory, dynamical systems, and mathematical physics. These publications sparked significant research activity, leading to various problems related to this new area. It is worth noting that evolution algebras are not defined by identities and do not belong to the varieties of nonassociative algebras like the Lie, Jordan, or alternative algebras. Therefore, the investigation of this type of algebra requires a different approach. The classification of a concrete class of algebras is always a severe problem. It was classified all Jordan algebra structures on a two-dimensional vector space over $\mathbb{R}$ in terms of their matrices of structure constants [1]. Low-dimensional evolution algebras have been classified [10]. It was classified all possible three-dimensional evolution algebra [3]. However, a complete classification of evolution algebras has yet to be achieved. Therefore, finding another more effective approach for classification processes, particularly in higher-dimensional evolution algebras, is crucial. In this work, we introduce the concept of exact sequences into evolution algebras to aid in solving the classification problem.

Here, we should note that most investigations have focused on nilpotent evolution algebras, as demonstrated by the work of Elduque and Labra [7], which introduced new properties of these algebras, and Casas et al. [4] explored their decompositions and classifications. Although only a few studies have focused on non-nilpotent evolution algebras, Casas et al. [2] explored evolution algebras of arbitrary dimension, while Celorrio and Velasco [6] classified algebras of dimensions two and three. It was investigated the Jacobson radical of evolution algebras [17] and Mukhamedov et al. [11] examined genetic algebras associated with permuted Lotka-Volterra operators. Additionally, Rozikov [14] discussed population dynamics in the context of evolution algebras. Genetic algebras, generated by quadratic stochastic operators, share a connection with evolution algebras in dynamical systems by modeling the evolution of interacting components, where non-associativity and stochasticity play a key role in understanding complex biological interactions and system dynamics, as explored in Embong et al. [9]. The work in Qaralleh and Mukhamedov [13] initiated an exploration of Volterra evolution algebras, which are closely linked to genetic Volterra algebras. Subsequently, Mukhamedov and Qaralleh [12] investigated S-evolution algebras, a broader class than Volterra algebras, addressing their solvability, simplicity, semisimplicity, and the structure of their enveloping algebras using corresponding graphs. For a comprehensive overview of recent advances in evolution algebras [5].

The exactness of sequences in evolution algebra can be used as a mathematical notion to capture the relationships and structures that evolution algebras involve. In evolution algebras, the exact sequences enable the decomposition and reconstruction of these algebras, allowing for a deeper analysis of their constituent components. This tool is particularly effective in decomposing algebraic structures in low-dimensional cases. It facilitates the scrutiny of fundamental genetic models and enables a more profound investigation of their underlying bases.

Extending the notion of exact sequences from low-dimensional to higher-dimensional evolution algebras would establish a general framework for uncovering richer interactions and hierarchical structures. As one takes on higher dimensions, the algebraic structures become more complex, with a growing number of generators and relations. It is at this stage that the use of exact sequences affords the ability to systematically study the ways in which these algebras evolve and transition in their fundamental properties and behaviours in higher-dimensional settings. This method also aids in mapping the arrangement of an evolution system's algebraic landscape while shedding light on the potential genetic mechanisms that these algebras model, thereby enhancing their theoretical and biological applications.

The theory of exact sequences in the context of non-associative algebras has received limited attention in the literature [8], leaving this field largely unexplored and presenting significant opportunities for further research. This paper is structured to explore both the theoretical background and applications of exact sequences in evolution algebras. We first introduce the basic concepts and definitions relevant to the evolution of algebras and exact sequences, providing the necessary preliminaries to understand the primary results. Next, we present the Hypothetical Five Lemma and its consequences for evolution algebras. Finally, we introduce a detailed exact sequence, progressing from general exact sequences to specific exact sequences for $S-$evolution algebras, where we demonstrate fundamental properties and provide proofs of key theorems. The paper concludes with a discussion of the broader implications of these exact sequences, their applications, and their impact on mathematical theory and practical settings.

2 Preliminaries

In this section, we review key definitions and auxiliary notations that will be used throughout the paper.

Definition 2.1. *Let $\mathcal{E}$ be a vector space over a field $\mathbb{F}$ with a binary operation $\cdot$ and a basis $A = \{f_1, f_2, \dots, f_n\}$. The triple $(\mathcal{E}, \cdot, A)$ is called an evolution algebra if the following conditions hold,*

$$
\begin{aligned}
f_i \cdot f_j &= 0, && \text{for } i \neq j, \\
f_i \cdot f_i &= \sum_{k=1}^{n} b_{ik} f_k, && \text{for each } i = 1, \dots, n.
\end{aligned}
$$

The set A is referred to as the natural basis of $\mathcal{E}$, and the matrix $B = (b_{ij})_{i,j=1}^{n}$ is known as the structural matrix of $\mathcal{E}$ associated with A.

It is evident that an evolution algebra is commutative. Moreover, the rank of A equals the dimension of the product space, i.e., $\operatorname{rank}(A) = \dim(\mathcal{E} \cdot \mathcal{E})$. Unless stated otherwise, the field $\mathbb{F}$ is algebraically closed and of characteristic zero.

Following [12], a matrix $B = (b_{ij})_{i,j=1}^{n}$ is called an S-matrix if:

(i) $b_{ii} = 0$ *for all* $i \in \overline{1, n}$.

(ii) $b_{ij} \neq 0$ *if and only if* $b_{ji} \neq 0$ *for* $i \neq j$.

An S-evolution algebra is an evolution algebra whose structural matrix takes the form of an S-matrix. Specifically, if the structural matrix B is skew-symmetric, then $\mathcal{E}$ is called a Lotka-Volterra (or Volterra) evolution algebra [13]. Lotka-Volterra algebras represent a particular subclass of $S-$evolution algebras.

An exact sequence of evolution algebras is a sequence of homomorphisms between the algebras such that the image of one homomorphism equals the kernel of the next one,

$$\cdots \xrightarrow{\phi_{n-2}} E_{n-1} \xrightarrow{\phi_{n-1}} E_n \xrightarrow{\phi_n} E_{n+1} \xrightarrow{\phi_{n+1}} \cdots$$

where for each n, Im$(\phi_n) =$ Ker(ϕ_{n+1}). It can also be denoted as,

$$\cdots \to E_{n-1} \to E_n \to E_{n+1} \to \cdots$$

Note: A short exact sequence of evolution algebras is a sequence of homomorphisms between the algebras of the form,

$$0 \to E \xrightarrow{\alpha} F \xrightarrow{\beta} G \to 0,$$

where α is injective, β is surjective, and the image of α is equal to the kernel of β. This means that $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$, and the sequence is exact at each position.

Remark 2.1. *The following applications are important to note:*

1. *The use of exact sequences in the classification of evolution algebras is a powerful algebraic tool for studying their structural properties. The exact sequences allow us to determine extensions, decompositions, and relationships between evolution algebras, thus giving a systematic method towards their classification. For instance, they can be utilized to examine how lower-dimensional algebras combine to form higher-dimensional algebraic structures or to establish the conditions under which an evolution algebra can be uniquely split into direct sums. Moreover,the exact sequence can also explains inheritance mechanisms and evolutionary processes by classifying algebras based on their structural properties.*

2. *The incorporation of the Splitting Lemma and the Five Lemma of (co)homological algebra into the investigation of evolutio algebras, and more specifically, symmetric evolution algebras, is due to their promise to reveal further intrinsic structural and categorical characteristics. The Splitting Lemma is important in establishing the conditions under which a short exact sequence of evolution algebras can be split, thereby allowing the decomposition of complicated algebras into direct sums of simpler constituents. This aspect is fundamental in the study of the structure of symmetric evolution algebras. The Five Lemma is also helpful in examining isomorphisms between commutative diagrams, with the aim of preserving structural properties and facilitating the classification of evolution algebras.*

Definition 2.2. *A short exact sequence $0 \to E \xrightarrow{\alpha} F \xrightarrow{\beta} G \to 0$ of evolution algebras is said to be split exact if there exists an evolution algebra homomorphism $s : G \to F$ such that the composition $G \xrightarrow{\beta} F \xrightarrow{s} G$ is the identity on G. In other words, F is an isomorphic to the direct product of evolution algebras $E \oplus G$.*

The following formulation follows the structure of the classical Five Lemma in homological algebra for modules, which connects properties of morphisms (monomorphisms and epimorphisms) in exact sequences.

Lemma 2.1. *Consider the following commutative diagram of evolution algebras and homomorphisms with the exact rows,*

$$\begin{array}{ccccccccc}
E_1 & \xrightarrow{\varphi_1} & E_2 & \xrightarrow{\varphi_2} & E_3 & \xrightarrow{\varphi_3} & E_4 & \xrightarrow{\varphi_4} & E_5 \\
\downarrow\alpha_1 & & \downarrow\alpha_2 & & \downarrow\alpha_3 & & \downarrow\alpha_4 & & \downarrow\alpha_5 \\
F_1 & \xrightarrow{\phi_1} & F_2 & \xrightarrow{\phi_2} & F_3 & \xrightarrow{\phi_3} & F_4 & \xrightarrow{\phi_4} & F_5
\end{array}$$

Then,

(1) *If α_2 and α_4 are epimorphisms and α_5 is a monomorphism then α_3 is an epimorphism.*

(2) *If α_2 and α_4 are monomorphisms and α_1 is an epimorphisms then α_3 is a monomorphisms.*

3 General Construction of Exact Sequences in Evolution Algebras

In this section, we are going to construct exact sequences of general evolution algebras. Let,

$$E = \langle e_i \mid 1 \leq i \leq N \rangle, \quad \text{and} \quad F = \langle f_j \mid 1 \leq j \leq M \rangle,$$

be evolution algebras. In Tian [15], it was shown that the direct product of two evolution algebras is itself evolution algebra. Therefore, $E \oplus F$ is an evolution algebra of dimension $N + M$.

Proposition 3.1. *Let $E = \langle e_i \mid 1 \leq i \leq N \rangle$ and $F = \langle f_j \mid 1 \leq j \leq M \rangle$ be evolution algebras. Then the mappings $\varphi : E \to E \oplus F$ and $\psi : E \oplus F \to F$, which are defined as follows,*

- *$\varphi(x) = (x, 0)$ for any $x \in E$,*
- *$\psi((x, y)) = y$ for any $x \in E$ and $y \in F$,*

are evolution algebra homomorphisms.

Proof. For the sake of simplicity, we only prove the first map, φ, and the proof for the second map, ψ, follows in the same manner. Let $x = \sum_{i=1}^{N} \alpha_i e_i$ and $y = \sum_{i=1}^{N} \beta_i e_i$ be two elements of E. In order to prove that φ is an evolution algebra homomorphism, we first need to show that φ is linear. However, it is obvious that φ is linear. Therefore, it is enough to show the multiplicative condition, i.e., that $\varphi(xy) = \varphi(x)\varphi(y)$. Consider,

$$\begin{aligned}
\varphi(xy) &= \varphi\left(\left(\sum_{i=1}^{N} \alpha_i e_i\right)\left(\sum_{i=1}^{N} \beta_i e_i\right)\right) = \varphi\left(\sum_{i=1}^{N} \alpha_i \beta_i e_i^2\right) \\
&= \sum_{i=1}^{N} \alpha_i \beta_i \varphi(e_i^2) = \sum_{i=1}^{N} \alpha_i \beta_i (e_i^2, 0) \\
&= \sum_{i=1}^{N} \alpha_i (e_i, 0) \beta_i (e_i, 0) = \left(\sum_{i=1}^{N} \alpha_i e_i, 0\right)\left(\sum_{i=1}^{N} \beta_i e_i, 0\right) \\
&= \varphi(x)\varphi(y).
\end{aligned}$$

Therefore, φ is an evolution algebra homomorphism. □

Theorem 3.1. *Given any two evolution algebras E and F, there exists an evolution algebra G and homomorphisms $\varphi : E \to G$ and $\psi : G \to F$ such that the sequence $E \to G \to F$ is exact.*

Proof. Let $E = \langle e_i \mid 1 \leq i \leq N \rangle$ and $F = \langle f_j \mid 1 \leq j \leq M \rangle$ be evolution algebras with structural matrices of constants $M_1 = (a_{ij})_{i,j=1}^{N}$ and $M_2 = (b_{ij})_{i,j=1}^{M}$, respectively.

We define G as the evolution algebra direct product of E and F, denoted by,

$$G = E \oplus F = \langle (e_i, 0), (0, f_j) \mid e_i \in E,\ f_j \in F \rangle,$$

where the structural matrix of constants for G is given by,

$$M = \begin{pmatrix} M_1 & \mathbf{0} \\ \mathbf{0} & M_2 \end{pmatrix},$$

where M is an $(N+M)\times(N+M)$. Now, G is an evolution algebra with $\dim(G) = N + M$.

We define the homomorphisms $\varphi : E \to G$ and $\psi : G \to F$ as follows,

$$\begin{aligned} \varphi(x) &= (x, 0) \qquad \text{for all } x \in E, \\ \psi((x, y)) &= y, \qquad \text{for all } (x, y) \in G. \end{aligned}$$

By Propositions 3.1, both φ and ψ are evolution algebra homomorphisms. Next, we are going to show that the sequence $E \to G \to F$ is exact.

First, we show that $\text{Im}(\varphi) \subseteq \text{Ker}(\psi)$. Let $\varphi(x) \in \text{Im}(\varphi)$ where $x = \sum_{i=1}^{N} \alpha_i e_i \in E$. Then,

$$\varphi\left(\sum_{i=1}^{N} \alpha_i e_i\right) = \sum_{i=1}^{N} \alpha_i \varphi(e_i) = \sum_{i=1}^{N} \alpha_i \left(e_i, 0\right) = \left(\sum_{i=1}^{n} \alpha_i e_i, 0\right).$$

Applying ψ to $\varphi\left(\sum_{i=1}^{N} \alpha_i e_i\right)$, we have $\psi\left(\varphi\left(\sum_{i=1}^{N} \alpha_i e_i\right)\right) = \psi\left(\left(\sum_{i=1}^{N} \alpha_i e_i, 0\right)\right) = 0$. Hence, $\varphi(x) \in \text{Ker}(\psi)$, which implies that $\text{Im}(\varphi) \subseteq \text{Ker}(\psi)$.

Next, we prove that $\text{Ker}(\psi) \subseteq \text{Im}(\varphi)$. Take an arbitrary element $(x, y) \in \text{Ker}(\psi)$ where $x \in E$ and $y \in F$. Then, (x, y) can be written as,

$$(x, y) = \left(\sum_{i=1}^{N} \alpha_i e_i, \sum_{j=1}^{M} \beta_j f_j\right) = \left(\sum_{i=1}^{N} \alpha_i e_i, 0\right) + \left(0, \sum_{j=1}^{M} \beta_j f_j\right).$$

This means that,

$$\psi(x, y) = \psi\left(\left(\sum_{i=1}^{N} \alpha_i e_i, 0\right) + \left(0, \sum_{j=1}^{M} \beta_j f_j\right)\right) = \psi\left(\sum_{i=1}^{N} \alpha_i e_i, 0\right) + \psi\left(0, \sum_{j=1}^{M} \beta_j f_j\right).$$

Then,

$$\psi(x, y) = \sum_{j=1}^{M} \beta_j f_j = 0.$$

Since the set $\{f_j : 1 \leq j \leq M\}$ is linearly independent, then we have $\beta_j = 0$ for any j. Therefore, $(x, y) = \left(\sum_{i=1}^{N} \alpha_i e_i, 0\right) \in \text{Im}(\varphi)$, and $\text{Ker}(\psi) \subseteq \text{Im}(\varphi)$.

Hence, we have shown that $\text{Im}(\varphi) = \text{Ker}(\psi)$, and the sequence $E \to G \to F$ is exact.

Therefore, given any two evolution algebras E and F, there exists an evolution algebra G and homomorphisms $\varphi : E \to G$ and $\psi : G \to F$ such that the sequence $E \to G \to F$ is exact. □

Now, let us consider the following short exact sequence of evolution algebras,

$$0 \longrightarrow E \xrightarrow{\varphi} G \xrightarrow{\psi} F \longrightarrow 0, \tag{1}$$

where the homomorphisms,

$$\varphi(x) = (x, 0), \qquad \psi(x, y) = y.$$

Corollary 3.1. *The short exact sequence of evolution algebras given by (1) is split. i,e., $G \cong E \oplus F$.*

Proof. Consider the short exact sequence $0 \longrightarrow E \xrightarrow{\varphi} G \xrightarrow{\psi} F \longrightarrow 0$ of evolution algebras. Define the homomorphism $\alpha : F \to G$ by,

$$\alpha\left(\sum_{j=1}^{M} \gamma_j f_j\right) = \left(0, \sum_{j=1}^{M} \gamma_j f_j\right).$$

Now, consider the composition,

$$(\psi \circ \alpha)\left(\sum_{j=1}^{M} \gamma_j f_j\right) = \psi\left(0, \sum_{j=1}^{M} \gamma_j f_j\right) = \sum_{j=1}^{M} \gamma_j f_j.$$

Thus, $\psi \circ \alpha = id_F$. Therefore, the given exact sequence of evolution algebra is split and $G \cong E \oplus F$. ☐

We now present an illustrative example of a short exact sequence of evolution algebras to demonstrate how the exactness helps verify the isomorphism between two evolution algebras.

Example 3.1. *Let $E_1 = \langle e_i \rangle_{i=1}^{2}$, $E_2 = \langle f_i \rangle_{i=1}^{4}$ and $E_3 = \langle g_i \rangle_{i=1}^{2}$ be evolution algebras with the following table of multiplication,*

$$\begin{array}{lll} e_1^2 = e_1, & e_2^2 = 0, & \\ f_1^2 = f_1 + f_2, & f_2^2 = f_4^2 = 0, & f_3^2 = f_3 + f_4, \\ g_1^2 = g_1, & g_2^2 = 0. & \end{array}$$

Consider $H := E_1 \oplus E_3$, which forms a new evolution algebra as follows,

$$(e_1, 0)^2 = (e_1, 0), \quad (e_2, 0)^2 = (0, g_2)^2 = (0, 0), \quad (0, g_1)^2 = (0, g_1).$$

Our aim is to show that $E_2 \cong H$. Now, one can see that the following sequence of evolution algebras is a short exact sequence,

$$0 \to E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \to 0,$$

where the homomorphisms α and β are defined by,

$$\begin{array}{lll} \alpha(e_1) = f_1 + f_2, & \alpha(e_2) = f_2, & \\ \beta(f_1) = \beta(f_2) = 0, & \beta(f_3) = g_1 - g_2, & \beta(f_4) = g_2. \end{array}$$

Next, we are going to show that the above sequence is split. Define the homomorphism $s : E_3 \to E_2$ by,

$$s(g_1) = f_3 + f_4, \quad s(g_2) = f_4.$$

Assume that $x \in E_3$, then $x = a_1 g_1 + a_2 g_2$. Consider,

$$\begin{aligned} (\beta \circ s)(x) &= \beta(a_1 s(g_1) + a_2 s(g_2)) \\ &= \beta(a_1(f_3 + f_4) + a_2 f_4) \\ &= a_1 \beta(f_3 + f_4) + a_2 \beta(f_4) \\ &= a_1(g_1 - g_2 + g_2) + a_2 g_2 \\ &= a_1 g_1 + a_2 g_2 \\ &= x. \end{aligned}$$

Thus, $\beta \circ s = id_{E_3}$, which implies that the above exact sequence is split. Therefore, $E_2 \cong E_1 \oplus E_3 = H$.

4 Exact Sequence of $S-$Evolution algebras

In this section, we need to construct the short exact sequence of $S-$evolution algebra whose structural matrix of constants is a symmetric matrix. Given an $S-$evolution algebra,

$$E(SS) = \langle g_i : 1 \leq i \leq K \rangle,$$

let $E^S = \langle e_i : 1 \leq i \leq N \rangle$ be an evolution subalgebra of $E(SS)$, and let $E_S = \langle f_i : 1 \leq i \leq M \rangle$ be an evolution subalgebra of $E(SS)$, where f_i is the extended basis of e_i, where the structural matrix of $E(SS)$ is given by,

$$A_{E(SS)} = \begin{bmatrix} \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{12} & 0 & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & \cdots & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{N\times M} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{M\times N} & \begin{bmatrix} 0 & b_{12} & \cdots & b_{1M} \\ b_{12} & 0 & \cdots & b_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1M} & b_{2M} & \cdots & 0 \end{bmatrix} \end{bmatrix}, \tag{2}$$

and the structural matrices of E^S and E_S are given, respectively, by,

$$A_{E^S} = \begin{bmatrix} 0 & c^{12} & \cdots & c^{1N} \\ c^{12} & 0 & \cdots & c^{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c^{1N} & c^{2N} & \cdots & 0 \end{bmatrix}, \tag{3}$$

and

$$A_{E_S} = \begin{bmatrix} 0 & c_{12} & \cdots & c_{1M} \\ c_{12} & 0 & \cdots & c_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1M} & c_{2M} & \cdots & 0 \end{bmatrix}. \tag{4}$$

In the following discussion, we assume that all entries in the structural constants of the matrices E^S and E_S are nonzero, except for the diagonal entries. Additionally, E^S and E_S are evolution subalgebras with the property that A_{E^S} and A_{E_S} are invertible. Furthermore, the nonzero matrices in $A_{E(SS)}$ also satisfy these properties.

This assumption is made to prevent the complexity of addressing multiple cases, which would otherwise complicate the calculations and detract from the clarity of the main results. By adopting this simplification, we preserve clarity and ensure a more manageable analysis, while still maintaining the generality and essential structural properties of the evolution algebras.

Assume that $\pi_1, \pi_2 \in S_K$, and that π_1, π_2 are disjoint cycles of order N and M, respectively.

Proposition 4.1. *Let $\varphi : E^S \to E(SS)$. Then, $\varphi(e_i) = \dfrac{c^{ij}}{a_{\pi_1(i)\pi(j)}} g_{\pi_1(i)}$ is a homomorphism if and only if,*

$$\frac{c^{ij}}{a_{\pi_1(i)\pi_1(j)}} = \frac{c^{km}}{a_{\pi_1(k)\pi_1(m)}},$$

for any $1 \leq i \neq j \leq N,\ 1 \leq k \neq m \leq N$.

Proof. The homomorphism is obtained by the following linear transformation,

$$e_i = \sum_{k=1}^{K} \alpha_{ik} g_k.$$

Then, from the fact that $e_i e_j = 0$ for any $i \neq j$ and the invertibility of A_E one has that,

$$e_i = \alpha_{i\pi_1(i)} g_{\pi_1(i)}.$$

Now, let us consider,

$$e_i^2 = \sum_{k=1}^{N} c^{ik} e_k = \sum_{k=1}^{N} c^{ik} \alpha_{k\pi_1(k)} g_{\pi_1(k)}. \tag{5}$$

On the other hand,

$$e_i^2 = \alpha_{i\pi_1(i)}^2 g_{\pi_1(i)}^2 = \alpha_{i\pi_1(i)}^2 \sum_{m=1}^{N} a_{\pi_1(i)m} g_m. \tag{6}$$

Assume that $\pi_1(j) = l_0$ and $\pi_1(i) = r_0$. Then, by comparing (5) and (6), we obtain the following equation,

$$c^{ij} \alpha_{jl_0} = a_{r_0 l_0} \alpha_{ir_0}^2.$$

and from e_j^2 we have the following equation,

$$c^{ji} \alpha_{ir_0} = a_{l_0 r_0} \alpha_{jl_0}^2.$$

By the last two equations, one gets $\alpha_{jl_0}^3 = \alpha_{ir_0}^3$, then, $\alpha_{i\pi_1(i)} = \alpha_{j\pi_1(j)}$. This means that,

$$\alpha_{i\pi_1(i)} = \alpha_{j\pi_1(j)} = \frac{c^{ij}}{a_{\pi_1(i)\pi_1(j)}},$$

but i and j are arbitrary, hence,

$$\alpha_{k\pi_1(k)} = \alpha_{m\pi_1(m)} = \frac{c^{km}}{a_{\pi_1(k)\pi_1(m)}}.$$

This implies that,

$$\frac{c^{ij}}{a_{\pi_1(i)\pi(j)}} = \frac{c^{km}}{a_{\pi_1(k)\pi_1(m)}},$$

for any $1 \leq i \neq j \leq N$, $1 \leq k \neq m \leq N$. The reverse part is straightforward. □

The following example illustrates the notations of above proposition.

Example 4.1. *Consider $E(SS) = \langle g_i : 1 \leq i \leq 5 \rangle$, $E^S = \langle e_1, e_2, e_3 \rangle$ and $E_S = \langle f_1, f_2 \rangle$. Then, their structural matrices of constants as follows,*

$$A_E(SS) = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 0 \\ a_{12} & 0 & a_{23} & 0 & 0 \\ a_{13} & a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{12} \\ 0 & 0 & 0 & b_{12} & 0 \end{pmatrix}, \; A_{E^S} = \begin{pmatrix} 0 & c^{12} & c^{13} \\ c^{12} & 0 & c^{23} \\ c^{13} & c^{23} & 0 \end{pmatrix}, \; A_{E_S} = \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{12} & 0 & c_{23} \\ c_{13} & c_{23} & 0 \end{pmatrix}.$$

Now, let $\pi_1 = (1)(2)(3) \in S_5$ and $\pi_2 = (4)(5) \in S_5$. Then, the mapping $\varphi : E^S \to E(SS)$ is defined by $\varphi(e_i) = \dfrac{c^{ij}}{a_{ij}} g_i$ is a homomorphism if and only if $\dfrac{c^{ij}}{a_{ij}} = \dfrac{c^{km}}{a_{km}}$.

The proof for the following two propositions is identical to the proof provided in Proposition 4.1. Therefore, it is omitted.

Proposition 4.2. *Let $\psi : E(SS) \to E_S$ then,*

$$\psi(g_i) = \begin{cases} \dfrac{b_{ij}}{c_{\pi_2(i)\pi_2(j)}} f_{\pi_2(i)}, & \text{if} \quad i \in \pi_2, \\ 0, & \text{if} \quad i \in \pi_1, \end{cases}$$

is a homomorphism if and only,

$$\frac{b_{ij}}{c_{\pi_2(i)\pi_2(j)}} = \frac{b_{km}}{b_{\pi_2(k)\pi_2(m)}}.$$

Proposition 4.3. *Let $S : E_S \to E(SS)$ then $S(f_i) = \dfrac{c_{ij}}{b_{\pi_2^{-1}(i)\pi_2^{-1}(j)}} g_{\pi_2^{-1}(i)}$ is a homomorphism if and only,*

$$\frac{c_{ij}}{b_{\pi_2^{-1}(i)\pi_2^{-1}(i)}} = \frac{c_{km}}{b_{\pi_2^{-1}(k)\pi_2^{-1}(m)}}.$$

In what follows, for convenience, we will choose,

$$\pi_1 = (1)(2)\dots(N) \in S_N, \qquad \text{and} \qquad \pi_2 = (N+1), (N+2), \dots, (N+M) \in S_M.$$

Theorem 4.1. *Let,*

$$E^S = \langle e_i : \ 1 \leq i \leq N \rangle, \qquad E_S = \langle f_j : 1 \leq j \leq M \rangle, \qquad \text{and} \qquad E(SS) = \langle g_m : 1 \leq m \leq K \rangle,$$

be $S-$evolution algebras, where the structural matrices of constants are given by (3), (4) and (2), respectively. Then, the sequence,

$$0 \longrightarrow E^S \xrightarrow{\varphi} E(SS) \xrightarrow{\psi} E_S \longrightarrow 0 \quad \text{is exact},$$

where φ is given in Proposition 4.1 and ψ is given in Proposition 4.2.

Proof. To show that the sequence is exact, we have to prove that φ and ψ are homomorphisms, φ is injective, ψ is surjective, and $\text{Im}(\varphi) = \text{Ker}(\psi)$.

φ, ψ are homomorphisms from Propositions 4.1 and 4.2. Now, to show that φ is injective it is enough to prove that $\text{Ker}(\varphi) = \{0\}$. Let $x \in E^S$, then $x = \sum\limits_{i \in \pi_1}^{N} \beta_i e_i$. Consider,

$$\varphi(x) = \sum_{i \in \pi_1}^{N} \beta_i \varphi(e_i) = \sum_{i \in \pi_1}^{N} \beta_i \frac{c^{ij}}{a_{\pi_1(i)\pi_1(j)}} g_{\pi_1(i)} = 0.$$

Since the set $\{g_{\pi_1(i)} \ : \ i \ \in \ \pi_1\}$ is linearly independent, it follows that $\beta_i \dfrac{c^{ij}}{a_{\pi_1(i)\pi_1(j)}} = 0$, but $\dfrac{c^{ij}}{a_{\pi_1(i)\pi_1(j)}} \neq 0$. Hence, $\beta_i = 0$, which implies that $x = 0$. Thus, φ is injective.

Now, we are going to show that ψ is surjective. Let $y = \sum_{i\in\pi_2}^{M} \gamma_i f_i \in E_S$. Then, choose $x = \sum_{i\in\pi_2}^{M} \gamma_i \frac{c_{\pi_2(i)\pi_2(j)}}{b_{ij}} g_{\pi_2^{-1}(i)}$ then,

$$\psi(x) = \sum_{i\in\pi_2}^{M} \gamma_i \frac{c_{\pi_2(i)\pi_2(j)}}{b_{ij}} \psi(g_{\pi_2^{-1}(i)}) = \sum_{i\in\pi_2}^{M} \gamma_i \frac{c_{\pi_2(i)\pi_2(j)}}{b_{ij}} \frac{b_{ij}}{c_{\pi_2(i)\pi_2(j)}} f_i = \sum_{i\in\pi_2}^{M} \gamma_i f_i.$$

This implies that $\psi(x) = y$. Hence, ψ is surjective.

Finally, we have to show that $\text{Im}(\varphi) = \text{Ker}(\psi)$. Let $x \in E^S$, then $\varphi(x) \in \text{Im}(\varphi)$. Then,

$$\varphi\left(\sum_{i\in\pi_1} \gamma_i e_i\right) = \sum_{i\in\pi_1} \gamma_i \varphi(e_i) = \sum_{i\in\pi_1} \gamma_i \frac{c^{ij}}{a_{\pi_1(i)\pi_1(j)}} g_{\pi_1(i)}.$$

Consider,

$$(\psi \circ \varphi)(x) = \sum_{i\in\pi_1} \gamma_i \frac{c^{ij}}{a_{\pi_1(i)\pi_1(j)}} \psi(g_{\pi_1(i)}) = 0.$$

This means that $\varphi(x) \in \text{Ker}(\psi)$. Hence, $\text{Im}(\varphi) \subseteq \text{Ker}(\psi)$. Assume that $y \in \text{Ker}(\psi)$ then,

$$\psi(y) = \psi\left(\sum_{i=1}^{K} \sigma_i g_i\right) = \sum_{i\in\pi_1} \sigma_i \psi(g_i) + \sum_{i\in\pi_2} \sigma_i \psi(g_i) = \sum_{i\in\pi_2} \sigma_i \psi(g_i) = 0,$$

but,

$$\sum_{i\in\pi_2} \sigma_i \psi(g_i) = \sum_{i\in\pi_2} \sigma_i \frac{b_{ij}}{c_{\pi_2(i)\pi_2(j)}} f_{\pi_2(i)} = 0.$$

Since the set $\{f_{\pi_2(i)}\}$ is linearly independent, we have $\sigma_i \frac{b_{ij}}{c_{\pi_2(i)\pi_2(j)}} = 0$, so $\sigma_i = 0$ for all $i \in \pi_2$. This implies that $y = \sum_{i\in\pi_1} \sigma_i g_i \in \text{Im}(\varphi)$. Hence, $\text{Ker}(\psi) \subseteq \text{Im}(\varphi)$. This completes the proof. $\square$

Theorem 4.2. *The exact sequence of evolution algebras given in Theorem 4.1 is split, i.e.,*

$$E(SS) \cong E^S \oplus E_S.$$

Proof. Define the homomorphism $\sigma : E_S \to E(SS)$ as in Proposition 4.3. Let $x \in E_S$, then $x = \sum_{i\in\pi_2} \gamma_i f_i$. Hence,

$$\sigma(x) = \sum_{i\in\pi_2} \gamma_i \sigma(f_i) = \sum_{i\in\pi_2} \gamma_i \frac{c_{ij}}{b_{\pi_2^{-1}(i)\pi_2^{-1}(j)}} g_{\pi_2^{-1}(i)}.$$

Consider,

$$(\psi \circ \sigma)(x) = \sum_{i\in\pi_2} \gamma_i \frac{c_{ij}}{b_{\pi_2^{-1}(i)\pi_2^{-1}(j)}} \psi(g_{\pi_2^{-1}(i)}) = \sum_{i\in\pi_2} \gamma_i \frac{c_{ij}}{b_{\pi_2^{-1}(i)\pi_2^{-1}(j)}} \frac{b_{\pi_2^{-1}(i)\pi_2^{-1}(j)}}{b_{\pi_2\pi_2^{-1}(i)\pi_2\pi_2^{-1}(j)}} f_{\pi_2\pi_2^{-1}(i)}.$$

Hence,

$$(\psi \circ \sigma)(x) = \sum_{i\in\pi_2} \gamma_i f_i = x.$$

Therefore, $(\psi \circ \sigma) = id$. Thus, $E(SS) \cong E^S \oplus E_S$. $\square$

Theorem 4.3. *Given an exact sequence of $S-$evolution algebras,*

$$0 \longrightarrow E^S \xrightarrow{\varphi} E(SS) \xrightarrow{\psi} E_S \longrightarrow 0.$$

Then, there exists an exact sequence of $S-$evolution algebras,

$$0 \longrightarrow E^S \xrightarrow{q} E^S \oplus E_S \xrightarrow{p} E_S \longrightarrow 0,$$

such that the following diagram commutes,

$$\begin{tikzcd} 0 \arrow[r] & E^S \arrow[r, "\varphi"] \arrow[d, equal] & E(SS) \arrow[r, "\psi"] \arrow[d, "\sigma"] & E_S \arrow[r] \arrow[d, equal] & 0, \\ 0 \arrow[r] & E^S \arrow[r, "q"] & E^S \oplus E_S \arrow[r, "p"] & E_S \arrow[r] & 0. \end{tikzcd}$$

Moreover, σ is isomorphism. where q is an injection map and p is a projection map.

Proof. Let,

$$0 \longrightarrow E^S \xrightarrow{\varphi} E(SS) \xrightarrow{\psi} E_S \longrightarrow 0,$$

be exact sequence of $S-$evolution algebras then it is split by Theorem 4.2. Then, there exists a homomorphism $\lambda : E_S \to E^S$ such that $\psi \circ \lambda = id$. We claim that $E(SS) = \text{Ker}(\psi) \oplus \lambda(E_S)$. Let $y \in E(SS)$ then we can write $y = (y - (\lambda \circ \psi(y)) - (\lambda \circ \psi(y))$. But $(\lambda \circ \psi(y)) \in \lambda(E_S)$ and

$$\psi(y - (\lambda \circ \psi(y)) = \psi(y) - (\psi \circ \lambda) \circ \psi(y) = \psi(y) - id \circ \psi(y) = 0.$$

Hence, $\psi(y) - (\psi \circ \lambda) \in \text{Ker}(\psi)$. Now, suppose that $y \in \text{Ker}(\psi) \cap \lambda(E_S)$ such that $y = \lambda(z)$ then $0 = \psi(y) = \psi(\lambda(z)) = z$, then $y = \lambda(z) = \lambda(0) = 0$. Hence, $\text{Ker}(\psi) \cap \lambda(E_S) = \{0\}$. But the sequence is exact ($\text{Ker}(\psi) = \text{Im}(\varphi)$). Hence, $E(SS) = \text{Im}(\varphi) \oplus \lambda(E_S)$. Then any $y \in E(SS)$ can be write as $y = \varphi(x) + \lambda(z)$ where $x \in E^S$ and $z \in E_S$.

Next, we are going to show that the diagram commutes. Define the homomorphisms,

$$q : E^S \to E^S \oplus E_S, \quad \text{and} \quad p : E^S \oplus E_S \to E_S,$$

by,

$$q\left(\sum_{i=1}^{N} \alpha e_i\right) = \left(\sum_{i=1}^{N} \alpha e_i, 0\right), \; p\left(\sum_{i=1}^{N} \alpha e_i, \sum_{j=1}^{M} \alpha f_j\right) = \sum_{j=1}^{M} \alpha f_j.$$

Define $\sigma : E(SS) \to E^S \oplus E_S$ as follows,

$$\sigma(g_i) = \begin{cases} \left(\dfrac{a_{ij}}{c^{\pi_1^{-1}(i)\pi_1^{-1}(j)}} e_{\pi_1^{-1}(i)}, 0\right), & \text{if } g_i \in \text{Im}(\varphi), \\ \left(0, \dfrac{b_{ij}}{c_{\pi_2^{-1}(i)\pi_2^{-1}(j)}} f_{\pi_2^{-1}(i)}\right), & \text{if } g_i \in \lambda(E_S). \end{cases}$$

Now, to prove that the diagram commutes, we have to show that $\sigma \circ \varphi = q$ and $p \circ \sigma = \psi$.

Let $x \in E^S$ then $\varphi(x) \in \text{Im}(\varphi)$. Now, consider,

$$\begin{aligned}(\sigma \circ \varphi)(x) &= \sigma\left(\sum_{i=1}^{N} \alpha_i \varphi(e_i)\right) = \sigma\left(\sum_{i=1}^{N} \alpha_i \frac{c^{ij}}{a_{\pi_1(i)\pi_1(j)}} g_{\pi_1(i)}\right) \\ &= \sum_{i=1}^{N} \alpha_i \frac{c^{ij}}{a_{\pi_1(i)\pi_1(j)}} \sigma(g_{\pi_1(i)}) = \sum_{i=1}^{N} \alpha_i \frac{c^{ij}}{a_{\pi_1(i)\pi_1(j)}} \left(\frac{a_{\pi_1(i)\pi_1(j)}}{c^{ij}} e_i, 0\right) \\ &= \left(\sum_{i=1}^{n} \alpha_i e_i, 0\right) = q(x).\end{aligned}$$

Next, we show that $p \circ \sigma = \psi$. Let $y \in E(SS)$ then $y = \varphi(x) + \lambda(z)$ where $x \in E^S$, and $z \in E_S$.

$$(p \circ \sigma)(\varphi(x) + \lambda(z)) = p \circ (\sigma \circ \varphi)(x) + (p \circ \sigma)(\lambda(z)) = p \circ q + (p \circ \sigma)(\lambda(z)),$$

but $p \circ q = 0$. Hence,

$$(p \circ \sigma)(\varphi(x) + \lambda(z)) = (p \circ \sigma)(\lambda(z)).$$

Now, $\psi(\varphi(x) + \lambda(z)) = \psi(\lambda(z))$. So, it is enough to show that $(p \circ \sigma)(\lambda(z)) = \psi(\lambda(z))$.

$$\begin{aligned}(p \circ \sigma)(\lambda(z)) &= (p \circ \sigma)\left(\lambda\left(\sum_{i=1}^{M} \beta_i f_i\right)\right) = p \circ \sum_{i=1}^{M} \beta_i \frac{c_{ij}}{b_{\pi_2^{-1}(i)\pi_2^{-1}(j)}} \sigma(g_{\pi_2^{-1}(i)}) \\ &= p \circ \sum_{i=1}^{M} \beta_i \frac{c_{ij}}{b_{\pi_2^{-1}(i)\pi_2^{-1}(j)}} \left(0, \frac{b_{\pi_2^{-1}(i)\pi_2^{-1}(j)}}{c_{ij}} f_{\pi_2^{-1}\pi_2^{-1}(i)}\right) \\ &= \sum_{i=1}^{M} \beta_i f_{\pi_2(i)} = \psi(\lambda(z)).\end{aligned}$$

Thus, the diagram commutes. From the short five lemma, one can show that σ is an isomorphism. $\square$

Now, assume that the following is an exact sequence of $S-$evolution algebras,

$$0 \longrightarrow E^S \xrightarrow{\varphi} E(SS) \xrightarrow{\psi} E_S \longrightarrow 0.$$

To determine whether a functor F preserves exactness, we need to check whether applying F to the sequence results in another exact sequence of evolution algebras. This can be expressed as,

$$0 \longrightarrow F(E^S) \xrightarrow{F(\varphi)} F(E(SS)) \xrightarrow{F(\psi)} F(E_S) \longrightarrow 0.$$

If this new sequence is exact, then the functor F preserves exactness. This means that the functor maps exact sequences to exact sequences, preserving the same relationships between evolution algebras and homomorphisms. Determining whether a functor preserves exactness is important in many areas of mathematics, as it helps us understand functor behaviour and its impact on structures and relationships within mathematical objects.

Theorem 4.4. *Let,*

$$0 \longrightarrow E^S \xrightarrow{\varphi} E(SS) \xrightarrow{\psi} E_S \longrightarrow 0,$$

be an exact sequence of $S-$evolution algebras. Given any evolution algebra F, then,

$$0 \longrightarrow F \otimes E^S \xrightarrow{id \otimes \varphi} F \otimes E(SS) \xrightarrow{id \otimes \psi} F \otimes E_S \longrightarrow 0,$$

is exact.

Proof. Assume that $F = \langle h_i : 1 \leq i \leq l \rangle$ is any evolution algebra. Now, we are going to prove that the sequence,

$$0 \longrightarrow F \otimes E^S \xrightarrow{id \otimes \varphi} F \otimes E(SS) \xrightarrow{id \otimes \psi} F \otimes E_S \longrightarrow 0,$$

is exact. The first step is to show that $id \otimes \varphi$ is injective, to do that let $x \in F \otimes E^S$, then,

$$x = \sum_{p=1}^{l} \sum_{i=1}^{N} \beta_{pi}(h_p \otimes e_i).$$

Consider,

$$id \otimes \varphi(x) = \sum_{p=1}^{l} \sum_{i=1}^{N} \beta_{pi}(h_p \otimes \varphi(e_i)) = 0.$$

This is equivalent to

$$id \otimes \varphi(x) = \sum_{p=1}^{l} \sum_{i=1}^{N} \beta_{pi} \frac{c^{ij}}{a_{\pi_1(i)\pi(j)}} (h_p \otimes g_{\pi_1(i)}) = 0.$$

Since the set $\{h_p \otimes g_{\pi_1(i)}\}$ is linearly independent, it follows that $\beta_{pi} \dfrac{c^{ij}}{a_{\pi_1(i)\pi(j)}} = 0$ but $\dfrac{c^{ij}}{a_{\pi_1(i)\pi(j)}} \neq 0$ for any $1 \leq i \neq j \leq N$. Hence, $\beta_{pi} = 0$ for all $1 \leq p \leq l$ and $1 \leq i \leq N$ which proves that $x = 0$. Therefore, $\text{Ker}(id \otimes \varphi) = \{0\}$. Thus, $id \otimes \varphi$ is injective.

Now, we are going to prove that $id \otimes \psi$ is surjective. Assume that $y \in F \otimes E_S$, then,

$$y = \sum_{p=1}^{l} \sum_{i=1}^{M} \gamma_{pi}(h_p \otimes f_i).$$

Now, define $x \in F \otimes E(SS)$ as follows,

$$\sum_{p=1}^{l} \sum_{i=1}^{M} \gamma_{pi} \frac{c_{\pi_2(i)\pi_2(i)}}{b_{ij}} (h_p \otimes g_{\pi_2^{-1}(i)}).$$

Then, by checking,

$$id \otimes \psi(x) = \sum_{p=1}^{l} \sum_{i=1}^{M} \gamma_{pi} \frac{c_{\pi_2(i)\pi_2(i)}}{b_{ij}} (h_p \otimes \psi(g_{\pi_2^{-1}(i)})),$$

which is equivalent to

$$id \otimes \psi(x) = \sum_{p=1}^{l} \sum_{i=1}^{M} \gamma_{pi} \frac{c_{\pi_2(i)\pi_2(i)}}{b_{ij}} \frac{b_{ij}}{c_{\pi_2(i)\pi_2(i)}} (h_p \otimes f_i) = y.$$

Thus, $id \otimes \psi$ is surjective.

Finally, we are going to show that $\text{Im}(id \otimes \varphi) = \text{Ker}(id \otimes \psi)$. Let $x \in F \otimes E^S$, then $(id \otimes \varphi)(x) \in \text{Im}(id \otimes \varphi)$. Consider,

$$id \otimes \varphi(x) = \sum_{p=1}^{l} \sum_{i=1}^{N} \beta_{pi}(h_p \otimes e_i) = \sum_{p=1}^{l} \sum_{i=1}^{N} \beta_{pi}(h_p \otimes \varphi(e_i)) = \sum_{p=1}^{l} \sum_{i=1}^{N} \beta_{pi} \frac{c^{ij}}{a_{\pi_1(i)\pi(j)}} (h_p \otimes g_{\pi_1(i)}).$$

Now consider,

$$(id \otimes \psi) \circ (id \otimes \varphi)(x) = \sum_{p=1}^{l} \sum_{i=1}^{N} \beta_{pi} \frac{c_{ij}}{a_{\pi_1(i)\pi(j)}} (h_p \otimes \psi(g_{\pi_1(i)})) = 0.$$

Hence, $id \otimes \varphi \in \text{Ker}(id \otimes \psi)$ which implies that $\text{Im}(id \otimes \varphi) \subseteq \text{Ker}(id \otimes \psi)$. On the other hand, Assume that $y \in \text{Ker}(id \otimes \psi)$, then,

$$(id \otimes \psi)(y) = (id \otimes \psi) \left(\sum_{p=1}^{l} \sum_{i=1}^{N} \beta_{pi} (h_p \otimes \psi(g_i)) \right) = 0.$$

The last expression can be rewritten as follows,

$$(id \otimes \psi)(y) = \sum_{p=1}^{l} \sum_{i \in \pi_1} \beta_{pi} (h_p \otimes \psi(g_i)) + \sum_{p=1}^{l} \sum_{i \in \pi_2} \beta_{pi} (h_p \otimes \psi(g_i)) = \sum_{p=1}^{l} \sum_{i \in \pi_2} \beta_{pi} (h_p \otimes \psi(g_i)) = 0.$$

This implies that,

$$\sum_{p=1}^{l} \sum_{i \in \pi_2} \beta_{pi} \frac{b_{ij}}{c_{\pi_2(i)\pi_2(j)}} (h_p \otimes f_{\pi_2(i)}) = 0.$$

Since the set $\{h_p \otimes f_{\pi_2(i)}\}$ is linearly independent, it follows that $\beta_{pi} = 0$ for any $i \in \pi_2$. Thus, we conclude that $y = \sum_{p=1}^{l} \sum_{i \in \pi_1} \beta_{pi} (h_p \otimes \psi(g_i)) \in \text{Im}(id \otimes \varphi)$. Thus, $\text{Ker}(id \otimes \psi) \subseteq \text{Im}(id \otimes \varphi)$, which completes the proof. □

5 Conclusions

In this paper, we have developed and examined exact sequences in the context of evolution algebras, focusing on their construction and properties. By presenting a general framework for constructing exact sequences and offering illustrative examples, we exhibited fundamental algebraic results, including the isomorphism theorem, the splitting Lemma, and the Five Lemma. In addition, we used these sequences on the class of $S-$evolution algebras, demonstrating their usefulness in investigating and analyzing the structural relationships and properties of these algebras. This research demonstrates that exact sequences provide potent means of decomposing and reconstructing evolution algebras, enabling systematic investigations of their basic properties and behaviours and providing significant insight into the genetic mechanisms modelled by these algebras.

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